Adaptive Source Localization Based Station Keeping of Autonomous Vehicles

Samet Güler, Member, IEEE, Barış Fidan, Senior Member, IEEE, Soura Dasgupta, Fellow, IEEE, Brian D.O. Anderson, Life Fellow, IEEE, and Iman Shames, Member, IEEE

Abstract—We study the problem of driving a mobile sensory agent to a target whose location is specified only in terms of the distances to a set of sensor stations or beacons. The beacon positions are known, but the agent can continuously measure its distances to them as well as its own position. This problem has two particular applications: (1) capturing a target signal source whose distances to the beacons are measured by these beacons and broadcasted to a surveillance agent, (2) merging a single agent to an autonomous multi-agent system so that the new agent is positioned at desired distances from the existing agents. The problem is solved using an adaptive control framework integrating a parameter estimator producing beacon location estimates, and an adaptive motion control law fed by these estimates to steer the agent toward the target. For location estimation, a least-squares adaptive law is used. The motion control law aims to minimize a convex cost function with unique minimizer at the target location, and is further augmented for persistence of excitation. Stability and convergence analysis is provided, as well as simulation results demonstrating performance and transient behavior.

Index Terms—Adaptive Control, Autonomous Robots, Cooperative Control, Sensor Networks, Optimization.

I. INTRODUCTION

STEERING a mobile agent to a specified target location autonomously is a research topic that has been considered extensively in the literature yielding important results [1]–[3], [8]. A derivative of this problem is the case where the target location is not known by the mobile agent; instead the mobile agent senses either the range or bearing or both via sensing or communication with the target. There may also be instances where the mobile agent cannot directly sense or communicate with the target. In such scenarios one may obtain range measurements to a set of sensor stations (beacons) whose distances to the target do not alter. In this regard, we consider the following problem given in abstract terms.

Problem I.1. Consider a mobile sensory agent $A$ with position $y \in \mathbb{R}^n$, $n \in \{2,3\}$, a set of $N$ sensor beacons $S \triangleq \{S_1, \cdots, S_N\}$, $N > n$, with unknown position $x_i \in \mathbb{R}^n$ for each $S_i$, and an unknown target position $y^* \in \mathbb{R}^n$. Assume that $S_i$ are not collinear for $n = 2$ and not coplanar for $n = 3$. Assume also that the beacon-target distances $d_i^* \triangleq \|y^* - x_i\|$, beacon-agent distances $d_i(t) \triangleq \|y(t) - x_i\|$, and self-position $y$ are available to $A$. The task is to define a control law to generate $\dot{y}(t)$ such that for any given initial position $y(0) = y_0$, $y(t)$ converges to $y^*$ asymptotically.

Two interpretations of Problem I.1 in real-life applications are as follows. The first is the scenario where there are $N$ sensor stations and a target signal source with unknown positions $x_i$ and $y^*$, respectively. A mobile sensory agent is desired to be steered to the target signal source autonomously, using the station-target distance measurements $d_i^*$, and the station-agent distance measurements $d_i(t)$. The second scenario is that given an autonomous multi-agent system $\mathcal{S}$ with agent positions $x_i$, the goal is to steer the mobile agent $A$ to $\mathcal{S}$ in a way that $A$ has distance $d_i^*$ to each agent $x_i$.

Since the target location to be reached is not directly known in the above scenarios, one needs to follow an adaptive control approach. In [3] and [4], an adaptive switching control approach is applied to solve the second interpretation of Problem I.1 for $N = 3$ and $n = 2$, without assuming the knowledge of the agent’s own position $y$. It is assumed in these works that the mobile agent is modeled by single integrator kinematics and there exist small errors on the parameters $d_i^*$ and the measurements $d_i(t)$. An analogue to this problem is considered in the localization literature where the objective is to estimate the unknown location of a signal source using distance measurements between the source and a set of sensors with known positions. [10] and [11] examine a set of gradient algorithms for this problem based on quartic distance cost functions, and the geometric location set of the target to guarantee their convergence to the actual target location. The geometric sets are established to be nontrivial ellipsoidal and polytopic regions surrounding the sensor beacons. In [7], the convexity issue that arises in [10] and [11] is resolved, and global convergence to the target is guaranteed by use of an alternative, convex cost function with the gradient algorithm.

Circumnavigation problem is by nature very close to Problem I.1 and studied for persistently drifting target situation in [15], and for non-holonomic vehicles in [5]. [15] integrates the concepts of localization and motion control using a non-convex cost function. Likewise, Problem I.1 looks similar to
the simultaneous localization and mapping (SLAM) problem which studies localizing an autonomous robot with unknown position in an unknown environment while generating a map of the environment concurrently [16]–[18]. However, there are significant differences between Problem I.1 and SLAM. In SLAM, the main purpose is to generate an approximate map of the environment using the agent-beacon distance measurements and a separate motion control algorithm drives the robot in parallel, while in Problem I.1, the main goal is to move the agent to a target implicitly defined by the beacons. A SLAM algorithm with its distinct methods may be independently employed in the localization part of Problem I.1, but it needs to be carefully integrated with a motion control law to solve Problem I.1 completely.

In this paper, we study Problem I.1 assuming that the dynamics of the mobile autonomous vehicle is modeled by a single integrator kinematics. Similar to [8], we partition the problem into two sub-problems: (i) localization of the beacons using range-only measurements, and (ii) motion control of the mobile agent. In the localization and the motion control parts, we employ least-squares (LS) based parameter estimation [9] and a gradient control law, respectively. Using the certainty-equivalence approach, we then combine the two algorithms and synthesize an indirect adaptive control algorithm to achieve the objective. The adaptive control scheme is designed systematically to address the issues that arise due to integration of the localization and motion control laws, e.g., cost function convexity and estimation convergence for the multiple beacon/agent setting, and adaptive control for convergence without persistence of excitation.

Our approach differs from [4] in that we assume N beacons with unknown positions and the position information of the vehicle with respect to some reference frame, while [4] solves the second interpretation of Problem I.1 for the three sensor case without the assumption that the position of the mobile agent is known to it. As an advantage of the current work over [4], we may show the modularity property following from the nature of the indirect adaptive algorithm. To put it differently, any localization algorithm that achieves asymptotic convergence of the sensor position estimates to the correct values can be employed in place of the LS parameter estimation algorithm, while one may consider the replacement of the gradient-based motion control law by another controller which uniquely defines the target point based on the sensor locations. Also, the goal in this paper is to force an agent to approach a target using only agent-beacon distance measurements, as opposed to [8] where agent-target distance measurements are available to a agent. On the other hand, [7], [9] are only concerned with localization or the estimation of the target’s position from distance measurements under specific assumptions on the agent trajectory whereas this paper studies integration of the localization and motion control laws. The aforementioned differences bring more complexity when closing the feedback loop in this paper compared to [7]–[9].

The remainder of the paper is organized as follows. In Section II, a gradient-based control law which minimizes a convex cost function is synthesized for the fictitious case as if the sensor locations are known. Then, in Section III, the localization algorithm used to localize the N beacon positions is presented. In Section IV, the localization algorithm and the motion control law are combined to derive the indirect adaptive control approach to solve Problem I.1. Formal stability and convergence analysis take place in that section as well. Section V demonstrates the performance of the proposed controller via simulations, and finally, Section VI offers conclusions.

II. CONTROL LAW DESIGN: KNOWN BEACON POSITIONS CASE

A. Problem Definition

Having the control and parameter identification parts of our design as specified in Section I, we use a certainty equivalence approach to solve Problem I.1. Accordingly, for control law construction, we first consider the fictitious case where the locations \( x_1, \ldots, x_N \in \mathbb{R}^n \) are perfectly known by the agent A. Following is the corresponding control problem statement for this fictitious case, which differs from Problem I.1 only in the assumption on the agent’s knowledge of sensor positions:

\textbf{Problem II.1.} Consider a mobile sensory agent \( A \) with position \( y \in \mathbb{R}^n \), \( n \in \{2, 3\} \), a set of \( N \) beacons \( S \triangleq \{S_1, \ldots, S_N\} \), \( N > n \), with known position \( x_i \in \mathbb{R}^n \) for each \( S_i \), and an unknown target position \( y^* \in \mathbb{R}^n \). Assume that \( S_i \) are not collinear for \( n = 2 \) and not coplanar for \( n = 3 \). Assume also that the beacon-target distances \( d_i^* \triangleq \|y^* - x_i\| \), beacon-agent distances \( d_i(t) \triangleq \|y(t) - x_i\| \), and self-position \( y \) are available to \( A \). The task is to define a control law to generate \( \dot{y}(t) \) such that for any given initial position \( y(0) = y_0 \), \( y(t) \) converges to \( y^* \) asymptotically.

![Fig. 1. Depiction of the scenario in Problem II.1 for a four-agent case](image)

In Fig. 1, the scenario described in Problem II.1 is depicted for a four beacon case. Two problems that are mathematically equivalent to the discrete-time version of Problem II.1 have been defined (within localization context instead of motion control) and solved in [10], [11]. In these works, the non-convex weighted cost function

\[ J(y) = \frac{1}{4} \sum_{i=1}^{N} \lambda_{J_i} (d_i^2 - d_i^2)^2, \]  

(II.1)

is used together with the gradient descent law to determine the ellipsoidal regions where practical localization of the signal...
source is achieved globally using only source-sensor distance measurements. In (II.1), \( \lambda_{ji} > 0 \) are certain design weights and each term \( \lambda_{ji} (d_i^2 - d_{ij}^2)^2 \) penalizes the difference between \( d_i \) and \( d_{ij}^2 \). The weights \( \lambda_{ji} \) can be chosen based on the any additional a priori information that may be available. For example, if it is known that certain \( d_i^2 \) information is more reliable than others, then the corresponding \( \lambda_{ji} \) can be chosen larger.

Here we use an approach similar to [10], [11], using an alternative convex cost function in place of (II.1) and applying the derivations in continuous time instead of discrete time, noting that one could also apply different techniques in the extremum seeking literature. It is established in [7] that the cost function

\[
J(y) = \frac{1}{2} \sum_{i=1}^{N-1} \left( (y - \xi_i)^\top e_i \right)^2, \tag{II.2}
\]

where

\[
e_i \triangleq x_i - x_N, \tag{II.3}
\]

\[
\xi_i \triangleq x_N + \alpha_i e_i, \quad \alpha_i \triangleq \frac{||e_i||^2 + d_{ij}^2}{2||e_i||^2}.
\]

is convex, and hence has a unique minimum, making it useful in many gradient search applications. The idea is based on defining the unique target point \( y^* \) as the intersection of certain lines instead of intersection of circles with origins at sensor positions, for the sake of convexity.

Consider the sensor localization setting in Fig. 2. Specifically, in the formulation (II.3), we choose \( x_N \) as the reference point and denote the vector traversed from the \( x_N \) to the \( x_i \) by \( e_i \). Let \( l_{iN} \) denote the line passing through the two intersection points of the two non-concentric circles \( C(x_i, d_i^2) \) and \( C(x_N, d_N^2) \), viz., the circle with center \( x_i \) and radius \( d_i^2 \) and the circle with center \( x_N \) and radius \( d_N^2 \), respectively. This line is called the radical axis of the circle pair \( C(x_i, d_i^2), C(x_N, d_N^2) \) [13], and has the property that any point \( p \) on \( l_{iN} \) has equal powers with respect to the circles \( C(x_i, d_i^2), C(x_N, d_N^2) \), i.e.,

\[
||p - x_i||^2 - d_i^2 = ||p - x_N||^2 - d_N^2. \tag{II.4}
\]

The intersection of the \( N - 1 \) radical axes \( l_1, \ldots, l_{N-1} \) is \( y^* \), which is the unique point satisfying

\[(y - \xi_i)^\top e_i = 0, \quad \forall i \in \{1, \ldots, N - 1\}. \tag{II.5}\]

Hence, we have the following lemma.

**Lemma II.1.** For the setting of Problem II.1, the cost function (II.2) is convex, with its unique global minimizer at \( y = y^* \) and global minimum \( J(y^*) = 0 \).

**Proof.** This result for the 2D case is already established in [7]. The 3D extension is straightforward observing that the cost function (II.2) with the vectors \( e_i, \xi_i \) of (II.3) is the same as in 2D and it carries the same properties as in the 2D case. Nevertheless, it is worth noting that in the 3D case, the radical axes are replaced by radical planes and the cost function (II.2) is minimized at the intersection of the \( N - 1 \) radical planes. \( \square \)

**Remark II.1.** [7] uses an alternative convention for pairing the beacons, where radical axes for circle pairs \( C(x_i, d_i^2), C(x_{i+1}, d_{i+1}^2) \) are used to uniquely determine the target location. This selection leads to \( e_i = x_{i+1} - x_i, \xi_i = x_i + a_i e_i \), where \( a_i = \frac{||e_i||^2 + d_i^2}{2||e_i||^2} \), as an alternative to (II.3). We emphasize that further alternative formulations exist. Leaving the optimal selection for the graph structures of the sensor set as a future work, we use the formulation (II.3) in this paper.

Observe that with the cost function (II.2), the gradient based control rule for agent velocity to reach \( y^* \) is given by

\[
\dot{y} = -\frac{\partial J(y)}{\partial y} = -\sum_{i=1}^{N-1} \left( (y - \xi_i)^\top e_i \right) e_i \tag{II.6}
\]

where

\[
A \triangleq \sum_{i=1}^{N-1} e_i e_i^\top = EE^\top, \quad b \triangleq \sum_{i=1}^{N-1} e_i e_i^\top \xi_i = EZ, \quad E \triangleq [e_1, \ldots, e_{N-1}], \quad Z \triangleq [e_1^\top \xi_1, \ldots, e_{N-1}^\top \xi_{N-1}]^\top.
\]

Note that the control law (II.6) requires only the knowledge of the beacon positions \( x_i \) and the desired distances \( d_i^2, i = 1, \cdots, N \).

**Proposition II.1.** The control law (II.6) with the parameters being defined in (II.3) solves Problem II.1.

**Proof.** Since the cost function (II.2) is convex, the only minimum of \( J \) is global. Hence, the vector \( y \) converges to the global minimum in the limit with the gradient rule (II.6) regardless of the initial condition. \( \square \)

**Corollary II.1.** The unique minimum of the quadratic cost function (II.2) is given by

\[y^* = A^{-1}b. \tag{II.8}\]

In (II.8), since we assume noncollinear and noncoplanar sensor positions, it follows from the geometry of the sensors that the matrix \( A \) is invertible.
B. Comparison with Other Cost Functions

The cost function (II.1) can be considered as an alternative to (II.2) and the corresponding gradient agent velocity control rule for employing (II.1) is found as

$$\dot{y} = -\frac{\partial J(y)}{\partial y} = \sum_{i=1}^{N} \lambda_i \left( d_i^2 - \|y - x_i\|^2 \right) (y - x_i). \tag{II.9}$$

It is established in [10], [11] that (II.9) leads to global convergence of $y$ to $y^*$, provided that $y^*$ is within a certain ellipsoidal region defined by $x_1, \ldots, x_N$. However, when $y^*$ is outside of this region, (II.1) has multiple stable local minima; and depending on the value of $y(0)$, $y$ may converge to one of the false stable minimizers instead of $y^*$. [10], [11], [14]. Hence, the control law (II.9) does not provide a global solution to Problem II.1 for certain $y^*$ settings.

III. ADAPTIVE SIGNAL SOURCE LOCALIZATION

If the mobile agent $A$ knows the sensor positions $x_i$, it will converge to the target point described by the unique minimum of the function (II.2) with the gradient control law (II.6). However, in the setting of Problem I.1, the sensor positions $x_i$ are unknown. The agent $A$ can only measure the beacon distances $d_i$. Accordingly, we propose use of a scheme that first estimates the (relative) sensor positions and then uses the produced estimates in place of the actual positions $x_1, \ldots, x_N$, applying the certainty equivalence principle [12]. We revisit the adaptive source localization algorithms covered in [6] and [10] to comply with this objective. A formal definition of the localization task is described in the following.

Problem III.1. Consider a set of beacons $S_1, \ldots, S_N$ with unknown positions $x_1, \ldots, x_N \in \mathbb{R}^n$, where $n \in \{2, 3\}$ and $N > n$, and a mobile sensor agent $A$ with known position $y \in \mathbb{R}^n$. Generate the beacon position estimates $\hat{x}_1, \ldots, \hat{x}_N$ using the agent’s own position $y$, and the agent-beacon distance measurements $d_i(t) = \|y(t) - x_i\|$, $i = 1, \ldots, N$.

Problem III.1 has been addressed and solved with adaptive estimation methods based on gradient and LS approaches in [6], [8], [9]. In [6], the problem (for a single beacon $S$) is solved for mobile sensory agent $A$ with single-integrator kinematics, using a gradient-based localization algorithm. In [6], both stationary and slowly drifting $S$ cases are studied. In [8], the agent not only localizes $S$, but also moves to reach $S$. We generate the beacon position estimates $\hat{x}_1, \ldots, \hat{x}_N$, employing $N$ identical estimators using the localization algorithm of [8], [9], where estimator $i$ is fed by $d_i$ for $i = 1, \ldots, N$.

A. The Localization Algorithm

We first revisit the LS based localization algorithm proposed in [9] in parallel to the gradient algorithm of [8] for estimating position $x_i$ of each $S_i$ in Problem III.1. We first assume that $x_i$ is constant. The linear parametric model developed in [8], [9] is based on the identity

$$\frac{1}{2} \frac{d}{dt} \left( \|y(t)\|^2 - d_i^2 \right) = x_i^T \dot{y}(t). \tag{III.1}$$

The implementation of a localization algorithm based on (III.1) would require generating the derivative of $d_i(t)$, however, this would bring some numerical problems especially when the noise level in measurements is high. Instead, we use the following filtered version of (III.1) as a parametric model:

$$z_i(t) \equiv x_i^T \phi_i(t), \tag{III.2}$$

$$z_i(t) = 1 + \frac{1}{2} (y^T(y(t) - d_i^2(t) \tag{III.3}$$

$$\phi_i(t) \equiv \dot{\phi}_i(t) = -\alpha \zeta_i(t) \tag{III.4}$$

where the notation $f_1(\cdot) = f_2(\cdot)$ for two functions $f_1, f_2$ indicates that there exist $M_1, M_2 > 0$ such that for all $t \geq 0$, $\|f_1(t) - f_2(t)\| \leq M_1 e^{-M_2 t}; \alpha > 0, \zeta_i(0) \in \mathbb{R}$, and $\varphi(0) \in \mathbb{R}^n$ are arbitrary design parameters.

We propose use of the following modified form of the localization algorithm proposed and analyzed in [9], for Problem III.1 (and for Problem I.1), considering parametrization (III.2): $\hat{x}_i(t) = \text{Proj}_{R,\hat{x}_i(t)}(P(t)(z_i(t) - \hat{x}_i(t)) \phi_i(t)),$ \tag{III.5}$$

$$\dot{\hat{P}}(t) = \begin{cases} \beta \dot{P}(t) - P(t) \phi_i(t) \phi_i^T(t) P(t), & \text{if } \lambda_{\max}(P(t)) \leq \rho_{\max}, \\ 0, & \text{otherwise}, \end{cases} \tag{III.6}$$

$$P(t^+) = P_0 = \rho_0 I, \tag{III.7}$$

with arbitrary initial estimate $\hat{x}_i(0)$ satisfying $\|\hat{x}_i(0)\| \leq R$ for a pre-set upper-bound $R$, where $\beta > 0$ is the constant forgetting factor, $P$ is the adaptive gain matrix with the initial condition $P(0) = P_0 = \rho_0 I$, and $t^*$ is the resetting time instant defined by the condition $\lambda_{\min}(P(t^*)) = \rho_{\min}$ for some design coefficients $\rho_0 > \rho_{\min} > 0$. The modified LS gain update rule (III.6) with resetting guarantees that $\rho_{\min} I \leq P(t) \leq \rho_{\max} I$ for all $t \geq 0$ [12]. In (III.5), $\text{Proj}_{R,\hat{x}_i(t)}$ stands for a parameter projection operator [12] used to satisfy $\|\hat{x}_i(t)\| \leq R$ for all $t$, and is defined for a given estimate vector $\hat{x}_i(t)$, raw adaptive law input $v$ (having the same dimension as $\theta$), and scalar threshold $M_\theta > 0$ by

$$\text{Proj}_{M_\theta, \theta}(v) = \begin{cases} v, & \text{if } \|\theta\| < M_\theta, \\ \theta + \frac{\theta^T \theta}{M_\theta^2} v, & \text{if } \|\theta\| = M_\theta \text{ and } v^T \theta \leq 0, \\ v - \frac{\theta^T \theta}{M_\theta^2} v, & \text{otherwise}. \end{cases} \tag{III.8}$$

Above and in the sequel, where we propose an alternative parametric model/estimation setting to reduce computational cost, we make the following assumption:

Assumption III.1. The target position $y^*$, the global beacon positions $x_i$, and the relative positions $e_i = x_i - x_N$ satisfy $\|y^*\| \leq R, \|x_i\| \leq R$ and $\|e_i\| \leq R$ for some known $R > 0$.

Depending on the accuracy of the a priori information, $R$ in Assumption III.1 can be selected arbitrarily large.

B. Direct Estimation of $e_i$

The localization algorithm (III.5) generates the estimates of the $N$ sensor positions. In this subsection, in order to reduce computational cost, we introduce an alternative parametric
model to generate the estimates of the relative positions $x_i = x_i - x_N$, $i \in \{1, \ldots, N - 1\}$. Evaluating (III.1) for $i = N$, one has

$$
\frac{1}{2} \frac{d}{dt} \left( \|y(t)\|^2 - (d_N(t))^2 \right) = x_N^T \dot{y}.
$$

(III.9)

For $i \in \{1, \ldots, N - 1\}$, subtracting (III.9) from (III.1) gives

$$
\frac{1}{2} \frac{d}{dt} \left( (d_N(t))^2 - (d_i(t))^2 \right) = e_i^T \dot{y}.
$$

(III.10)

Analogously to Section III-A, we use the following filtered version of (III.10) as the parametric model:

$$
z_i(t) = e_i^T \phi(t),
$$

(III.11)

$$
z_i(t) \triangleq \xi_i(t) = -\alpha \xi_i(t) + \frac{1}{2} \left( (d_N(t))^2 - (d_i(t))^2 \right),
$$

(III.12)

where $\phi$ is defined in (III.4).

Based on Assumption III.1, we propose use of the following localization algorithm for estimation of $e_i$ based on the parametric model (III.11):

$$
\dot{\hat{e}}_i(t) = \text{Proj}_{R, \hat{e}_i(t)} \left( P(t) (z_i(t) - \hat{z}_i(t)) \phi(t) \right),
$$

(III.13)

$$
\dot{\hat{z}}_i(t) = \hat{e}_i(t) \phi(t),
$$

(III.14)

$$
\dot{x}_N(t) = \text{Proj}_{R, \hat{x}_N(t)} \left( P(t) (z_N(t) - \hat{x}_N(t)) \phi(t) \right).
$$

(III.15)

Based on the estimates $\hat{e}_i$ and $\hat{x}_N$ generated by (III.13), (III.16), we set the estimates $\hat{A}, \hat{b}$ of $A, b$ in (II.8) as follows:

$$
\hat{A} = \begin{cases} \hat{E} \hat{E}^T, & \text{if } \lambda_{\min}(\hat{E} \hat{E}^T) > \varepsilon_A \\ \hat{E} \hat{E}^T + (\varepsilon_A - \lambda_{\min}(\hat{E} \hat{E}^T)) I, & \text{otherwise} \end{cases}
$$

(III.17)

$$
\hat{b} = \hat{E} \hat{Z}
$$

(III.18)

$$
\hat{E} = [\hat{e}_1, \ldots, \hat{e}_{N-1}], \quad \hat{Z} = [\hat{e}_1 \hat{\xi}_1, \ldots, \hat{e}_{N-1} \hat{\xi}_{N-1}]^T
$$

$$
\hat{\xi}_i = \hat{x}_N + \hat{a}_i \hat{e}_i
$$

$$
\hat{a}_i = \frac{\|\hat{e}_i\|^2 + d_i^2 - d_i^2}{2\|\hat{e}_i\|^2}.
$$

where $\varepsilon_A > 0$ is a small design constant. Noting that $\hat{E} \hat{E}^T$ is positive semi-definite, the switching law (III.17) guarantees that $\hat{A}$ is a continuous function of time, and $\hat{A}(t)$ is positive definite with minimum eigenvalue $\lambda_{\min}(\hat{A}(t)) \geq \varepsilon_A$ for all $t \geq 0$. The system matrix estimates $\hat{A}$ and $\hat{b}$ in (III.18) will be used in Section IV-A for dynamic reference trajectory generation within the adaptive control scheme to be designed.

C. Stability and Convergence of the Localization Algorithm

In this subsection, we present the stability and convergence properties of the localization algorithm of Section III-B, which will later be used in determining the control laws and design of the overall adaptive control scheme. We denote the filtering errors by

$$
\delta_i(t) \triangleq e_i^T \phi(t) - z_i(t),
$$

(III.19)

$$
\delta_N(t) \triangleq x_N^T \phi(t) - x_N(t),
$$

(III.20)

and the estimation errors by

$$
\tilde{e}_i(t) \triangleq \hat{e}_i(t) - e_i,
$$

(III.21)

$$
\tilde{x}_N(t) \triangleq \hat{x}_N(t) - x_N.
$$

(III.22)

Convergence properties of the localization algorithm are stated in the following lemma.

Lemma III.1. Consider the localization algorithm (III.13), (III.16) with the parametric models (III.11), (III.14). Then the following properties hold:

i) $\delta_i(t) = -\alpha \delta_i(t)$, and hence $\delta_i(t)$ exponentially converges to zero for $i = 1, \ldots, N$.

ii) $\tilde{e}_i(t) = \text{Proj}_{R, \tilde{e}_i(t)} \left( -P(t) \phi(t) \phi^T(t) \tilde{e}_i(t) - P(t) \phi(t) \delta_i(t) \right)$

for $i = 1, \ldots, N - 1$, and $\tilde{x}_N(t) = \text{Proj}_{R, \tilde{x}_N(t)} \left( -P(t) \phi(t) \phi^T(t) \tilde{x}_N(t) - P(t) \phi(t) \delta_N(t) \right)$.

iii) For all $t \geq 0$, $P(t)$ is symmetric positive definite and satisfies $\rho_{\min} I \leq P(t) \leq \rho_{\max} I$ where $\rho_{\min}$ and $\rho_{\max}$ are as in (III.5).

iv) The error signals $\tilde{e}_i$, $\tilde{x}_N$, and the estimates $\hat{e}_i$, $\hat{x}_N$, $\hat{A}$, $\hat{b}$ are bounded.

v) The $n \times n$ matrix $\hat{A}(t)$ ($\forall t \geq 0$) is symmetric positive definite.

vi) If the regressor signal $\phi$ is persistently exciting, i.e. if for every $t \geq 0$ and for some $T_0 > 0$ there exist some constants $\alpha_1$, $\alpha_2$ such that

$$
\alpha_1 I \leq \int_t^{t+T_0} \phi(\tau) \phi(\tau)^T d\tau \leq \alpha_2 I,
$$

(III.23)

then $\tilde{e}_i(t)$ ($i = 1, \ldots, N - 1$) and $\tilde{x}_N(t)$, and hence $\hat{A} \triangleq A - \hat{A}$ and $\hat{b} \triangleq b - \hat{b}$ converge to zero exponentially as $t \to \infty$.

Proof. In [9], (i),(ii), and (vi) are proven for the case of estimating the vector $x_i$ by a single estimator. We invoke the result of [9] and state that those properties of the localization algorithm also hold here for $e_i$ for all $i \in \{1, \ldots, N - 1\}$ and $x_N$ without loss of generality, and noting that the introduced modifications do not affect these properties. (iii) is a direct corollary of the initialization $P_0 = \rho_0 I$, the symmetry preservation property of the LS algorithm (III.5), and the bounding modification and covariance reseting used in this algorithm. Parameter projection guarantees (iv), (v) follows by definition of $\hat{A}$ in (III.17).

Remark III.1. By (III.4), the relation between the signals $\phi$ and $\dot{y}$ is given by:

$$
\dot{\phi}(t) = -\alpha \phi(t) + \dot{y}(t).
$$
Note that the persistence of excitation condition (III.23) on \( \phi \) also requires the persistence of excitation of \( \dot{y} \). This can be interpreted as the signal \( \dot{y} \) should persistently span \( \mathbb{R}^n \), \( n = \{2, 3\} \), and avoid linear trajectories in \( \mathbb{R}^2 \) and planar trajectories in \( \mathbb{R}^3 \). However, this is contrary to the main goal which is reaching the target and stopping. We deal with this issue in the next section.

IV. ADAPTIVE MOTION CONTROL FOR UNKNOWN BEACON POSITIONS

In this section, we combine the motion control law of Section II and the parameter estimation (or localization) algorithm of Section III-B to constitute the adaptive controller as shown in Fig. 3 and propose a systematic solution for Problem 1.1.

A. The Adaptive Control Scheme

Note that the control law (II.6) requires only the signals \( e_i, i = 1, \cdots, N-1 \), and \( x_N \), and the distance measurements \( d_i, i = 1, \cdots, N \). We design our adaptive control scheme based on this control law, Proposition II.1, and certainty equivalence principle; replacing the signals \( e_1, \cdots, e_{N-1}, x_N \) by their estimates generated by the localization algorithm in Section III-B. We define a new (estimate) cost function, which can be defined in terms of these estimates, as

\[
J_{\text{est}}(\dot{y}) = \frac{1}{2} \sum_{i=1}^{N-1} \left( y - \hat{\xi}_i \right)^T \hat{\xi}_i.
\]

Based on (IV.1), we generate the following trajectory, starting at an arbitrary initial location \( \dot{y}(0) = \dot{y}_0 \):

\[
\dot{\dot{y}}(t) = -\frac{\partial J_{\text{est}}(\dot{y})}{\partial \dot{y}} = -\hat{A}(t)\dot{y}(t) + \hat{b}(t),
\]

where \( \hat{A} \) and \( \hat{b} \) are defined in (III.17)-(III.18).

The aim in the remainder of the design will be forming a control law which will generate \( \dot{y} \) such that (i) \( y \) and \( \dot{y} \) are bounded; (ii) the persistence of excitation condition in (III.23) is satisfied, and hence exponential convergence of \( \hat{A} \) and \( \hat{b} \) to \( A \) and \( b \), respectively, is guaranteed; and (iii) \( y \) asymptotically tracks \( \hat{y} \). To meet all of these requirements, we propose the following control law:

\[
\dot{y}(t) = \dot{y}(t) - \kappa(y(t) - \hat{y}(t)) + f(||D^* - D(t)||_\infty)\sigma(t)
\]

\[
\dot{\sigma}(t) = H(t)\sigma(t),
\]

where \( D^* = [d_1^*, \cdots, d_N^*]^T \), \( D = [d_1, \cdots, d_N]^T \), \( \kappa > 0 \) is a design constant. The auxiliary signal \( \sigma \) in the control law (IV.3)-(IV.4) is introduced for the purpose of the persistence of excitation of \( \dot{y}(t) \), being motivated by the design procedure of [8] and [15], where the function \( f \) and the matrix \( H \) are selected to satisfy the following assumptions.

**Assumption IV.1.** \( f : [0, \infty) \to [0, \infty) \) is a strictly increasing, differentiable, bounded function that satisfies \( f(0) = 0 \) and \( f(D) \leq D, \forall D > 0 \).

**Assumption IV.2.**

i) There exists \( a T > 0 \) such that for all \( t \geq 0 \)

\[
H(t + T) = H(t).
\]

ii) \( H(t) \) is skew-symmetric for all \( t \) and differentiable everywhere.

iii) The derivative of \( \sigma \) in (IV.4) is persistently exciting for any nonzero value of \( \sigma(0) \), that is, there exist positive \( T_1, \alpha_\sigma \) such that for all \( t \geq 0 \) there holds

\[
\alpha_\sigma \| \sigma(0) \|^2 \leq \int_{t}^{t+T_1} \| J_{\sigma}(\sigma) \| \parallel \hat{\sigma}(\tau) \parallel d\tau \leq \alpha_2 \| \sigma(0) \|^2 I.
\]

iv) For every \( \theta \in \mathbb{R}^n \), and every \( t \in [0, \infty) \), there exists a time instant \( t_1(t, \theta) \in [t, t+T_1] \) such that \( \theta^T \hat{\sigma}(t_1) = 0 \).

Some matrices \( H \) satisfying Assumption IV.2 are given in [8] and [15] for both 2D and 3D cases. For instance, in 2D, the matrix \( H \in \mathbb{R}^{2 \times 2} \) can be chosen in the form of

\[
H = h J, \quad h \in \{\mathbb{R} \setminus \{0\}\}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

A constant 3 by 3 matrix cannot however be found which satisfies the requirements of Assumption IV.2. In [8] and [15], the following switching matrix is given as an example to matrices satisfying Assumption IV.2 in 3D:

\[
H(t) = \begin{cases} \begin{bmatrix} g \left( r_{T_1}(t) \right) \\ H_1 \\ \left(1 - g \left( r_{T_1}(t) - T_2 \right) / \rho \right) H_2 \end{bmatrix}, & \text{if } 0 \leq r_{T_1}(t) \leq T_1, \\ \left(1 - g \left( r_{T_1}(t) - T_3 \right) / \rho \right) H_2, & \text{if } T_1 \leq r_{T_1}(t) \leq T_2, \\ \left(1 - g \left( r_{T_1}(t) - T_4 \right) / \rho \right) H_2, & \text{if } T_2 \leq r_{T_1}(t) \leq T_3, \\ \left(1 - g \left( r_{T_1}(t) - T_5 \right) / \rho \right) H_2, & \text{if } T_3 \leq r_{T_1}(t) \leq T_4, \\ \left(1 - g \left( r_{T_1}(t) - T_6 \right) / \rho \right) H_2, & \text{if } T_4 \leq r_{T_1}(t) \leq T_5, \end{cases}
\]

Here, for a suitably small \( \rho > 0 \),

\[
T_1 = \rho, \quad T_2 = \rho + \frac{\pi}{|h_1|}, \quad T_3 = 2\rho + \frac{\pi}{|h_1|},
\]

\[
T_4 = 3\rho + \frac{\pi}{|h_1|}, \quad T_5 = 3\rho + \frac{\pi}{|h_1|} + \frac{\pi}{|h_2|},
\]

\[
T_1 = T_6 = 4\rho + \frac{\pi}{|h_1|} + \frac{\pi}{|h_2|},
\]

and \( r_{T_1}(t) = t - K_{T_1}(t)T_1 \) where \( K_{T_1}(t) \) is the largest integer \( k \) satisfying \( t \geq kT_1 \),

\[
H_1 = \begin{bmatrix} 0 & 0 \\ 0 & h_1 J \end{bmatrix}, \quad H_2 = \begin{bmatrix} h_2 J & 0 \\ 0 & 0 \end{bmatrix}.
\]
$h_1$, $h_2$ are real nonzero scalars, $J$ is as in (IV.6), and
$$g(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}(1 - \cos(\pi t)), & 0 \leq t \leq 1 \\ 1, & t > 1. \end{cases}$$

**Lemma IV.1.** Consider $\sigma(t) : \mathbb{R} \to \mathbb{R}^n$, $n = \{2, 3\}$ in (IV.4), with $H(t) : \mathbb{R} \to \mathbb{R}^{\alpha \times n}$, $n = \{2, 3\}$ and $T_1$ defined as follows:
- For $n = 2$, $H(t)$ is defined as in (IV.6) with $T_1 = \frac{\pi}{h}$.
- For $n = 3$, $H(t)$ is defined as in (IV.7) with $T_1$ as in (IV.8).

Then, $\sigma(t)$ obeys Assumption IV.2.

**Proof.** The result is a direct corollary of Theorem 5.1 and Theorem 5.2 of [8], and Section 8 of [15].

We have the following lemma.

**Lemma IV.2 ([8]).** Consider (IV.4) with $\sigma : \mathbb{R} \to \mathbb{R}^n$ and $H(t) : \mathbb{R} \to \mathbb{R}^{\alpha \times n}$, $n = \{2, 3\}$, satisfying Assumption IV.2.

Then,
1. $\|\sigma(t)\| = \|\sigma(0)\|$ for all $t$.
2. There exists a finite constant $\bar{\sigma}$ such that $\|\hat{\sigma}(t)\| \leq \bar{\sigma}$ for all $t$.

**Assumption IV.3.** $\kappa$ is such that $\kappa > \bar{\sigma}$.

**B. Stability and Convergence.**

In Section III-C, we have established that the signals $\tilde{x_N}, \hat{e}_i, i \in \{1, \ldots, N-1\}$, and $\hat{e}_i, \delta_i$ for all $i$ are bounded. In this subsection, we show that the output vector $y$ and the regressor signal $\phi$ are also bounded, and prove the convergence of the agent to the target location.

Before stating the main stability and convergence result, observe that for the target tracking error
$$\varepsilon_y(t) \triangleq y(t) - y^*, \quad (IV.9)$$
by applying the triangular inequality to the sides of the triangle with vertices $y(t), y^*, x_i$, one has $\|\varepsilon_y(t)\| + d_i^* \geq d_i(t)$ and $\|\varepsilon_y(t)\| + d_i(t) \geq d_i^*$ for all $i \in \{1, \ldots, N\}$. Thus,
$$\|\varepsilon_y(t)\| \geq \max_{i \in \{1, \ldots, N\}} |d_i^* - d_i(t)| = \|D^* - D(t)\|_{\infty}. \quad (IV.10)$$

**Theorem IV.1.** Consider Problem 1.1 and use of the adaptive control scheme composed of the localization algorithm (III.13), (III.16); the trajectory generation law (IV.2); and the control law (IV.3),(IV.4) to solve this problem. Let Assumptions III.1, IV.1, IV.2, IV.3 hold. Furthermore assume that there exists a time instant $t_{rm} > 0$ after which no covariance resetting occurs in update of $P(t)$ in (III.13), (III.16). Then:

1. All the closed loop signals, including $\hat{y}, y, \hat{y}, \hat{y}, \phi$, are bounded.
2. $y(t)$ converges to $y^*$ asymptotically.

**Proof.** See Appendix A.

**Remark IV.1.** The proposed algorithm runs on the autonomous agent $A$ locally and uses sensing information obtained from only a sub-class of the overall multi-agent system, which typically corresponds to $N < 10$.

**V. Simulations.**

In this section, we present the simulation results of the adaptive control system synthesized in Section IV, combination of the gradient control rule of Section II and the localization algorithm of Section III. We assume there is a set $\mathcal{S} = \{S_1, S_2, S_3\}$ of three stationary sensors which are positioned as follows:
$$x_1 = [2, 0]^T, \quad x_2 = [10, 0]^T, \quad x_3 = [6, 6]^T.$$

For all simulations, the estimation parameters are chosen as follows:
$$\beta = 0.9, \quad \rho_0 = 10, \quad \rho_{\min} = 0.001, \quad \rho_{\max} = 100, \quad R = 5000, \quad \epsilon_A = 0.01.$$

We choose $f(x) = x$. In what follows, we simulate the system for different cases of system and controller parameters. For all cases, we show the motion of the mobile agent, the distance error $\|y - y^*\|$ between the mobile agent and the target, and the distance errors between the estimated parameters and their actual values, $\|y - \hat{y}\|, \|\epsilon_i - \hat{\epsilon}_i\|, \ (i = 1, 2)$, and $\|x_3 - \hat{x}_3\|.

**A. Case 1**

In this scenario, we assume the target is at the interior of the convex hull that the beacons constitute by letting $y^* = [8, 2]^T$, resulting in the distance values $d_1^* = 6.3246, d_2^* = 2.8284, d_3^* = 4.4721$ between the target and the sensors. The initial estimation parameter values are chosen as follows:
$$\hat{x}_1(0) = [2, 3]^T, \quad \hat{x}_2(0) = [2, 3]^T, \quad \hat{x}_3(0) = [2, 1]^T, \quad y(0) = [10, 20]^T.$$

The design coefficients are chosen as follows:
$$\kappa = 10, \quad \sigma(0) = [2, 0]^T, \quad H = J,$$
where $J$ is as in (IV.6). With these values, we get the responses in Figures 4 and 5. It is clearly observed from these results that the mobile agent converges to the target location with the proposed control algorithm asymptotically.

**B. Case 2**

In order to present the global convergence property of the proposed control law, we now assume that the target is located outside the convex hull that the beacons constitute by letting $y^* = [7, 2]^T$. We also change the initial location of the agent to $y(0) = [-10, 20]^T$. The other variable values are kept the same as in Case 1. The results for this setting are illustrated in Figures 6 and 7. Since the proposed convergence result is global, the mobile agent converges to the target regardless of the initial location of the agent or the target’s location.

**C. Case 3**

In this case, we simulate the system for $\kappa = 50$ by letting all other variable values the same as in Case 1. The results are shown in Figure 8 and 9. As $\kappa$ determines how the position $y$ of the agent converges to its estimate $\hat{y}$, increasing this coefficient fastens the convergence of the distance $\|y - \hat{y}\|$ to zero.
Fig. 4. Case 1: No noise, non-collinear sensor positions, $\kappa = 10$, $H = J$. Agent motion and the sensor positions.

Fig. 5. Case 1: No noise, non-collinear sensor positions, $\kappa = 10$, $H = J$. The agent converges to the target point asymptotically.

D. Case 4

Here, we keep the variable values the same as in Case 1 except for $H = 5J$ and $\kappa = 12$. In Figures 10 and 11, the simulation results are depicted. As expected, this increased the oscillations on the motion of the mobile agent in both $x$-axis and $y$-axis, thus resulting in circular paths with diminishing radius on the $x$-$y$ graph of the motion. As the control law (IV.3) suggests, since the magnitude of the $f$ function diminishes with time, the effect of the oscillations caused by the $\sigma$ term disappears asymptotically, therefore the mobile agent converges to the target. On the other hand, increasing $\kappa$ would reduce the oscillatory behavior.

E. Case 5

This scenario tests the robustness of the proposed control law to noise. We assume all coefficient values are as in Case 1, and the distance measurements $d_i(t)$ for all $i = \{1, 2, 3\}$ are corrupted by a zero-mean Gaussian noise such that $d_{\text{cor}}(t) = d_i(t) + X(t)$, where $X(t) \sim \mathcal{N}(0, 0.05)$. The results are depicted in Figures 12 and 13. Even though noise adds small ripples to the motion of the agent, the agent eventually converges to a neighborhood of the target asymptotically.

F. Case 6

In Problem I.1 of Section II, we assume non-collinear beacons on $\mathbb{R}^2$ or non-coplanar beacons on $\mathbb{R}^3$. The best scenario in which this assumption is satisfied for a three-agent case is when the beacons constitute an equilateral triangle, and
in the simulations so far we let the three sensors form a triangle close to an equilateral one. In this scenario, we consider an “almost” collinear beacons case under noisy measurements. Note that such situations naturally arise when the beacons are almost collinear in 2D or almost coplanar in 3D, which would be given as the problem setup and is completely independent of the autonomous agent control design.

We assume the target is at $y^* = [12, 0.5]^T$ and the beacons are positioned at the following locations:

$$x_1 = [2, 0]^T, \ x_2 = [9, 1]^T, \ x_3 = [16, 0]^T.$$ We keep all other coefficient values the same as in Case 1 and add measurement noise to $d_i(t)$ for all $i = \{1, 2, 3\}$ as in Case 5. The results are depicted in Figure 14 and 15. The distance error in the transient part of the motion is large compared to the previous cases, and the terms $e_i$ ($i = 1, 2$) do not converge to their actual values. One reason for this transient behavior is that $e_1$ and $e_2$ are almost parallel, and hence, the matrix $A$ in (2.7) is close to a singular one. The proposed switching algorithm (III.17) avoids the estimate matrix $\hat{A}(t)$ from hitting a singular one even under the almost collinear/coplanar beacons case.

G. Case 7

Another important factor on the agent’s path is the initial value of the estimate $\hat{y}$. In this scenario, we keep the variable values the same as in Case 1 except for $\hat{y}(0) = [11, 4]^T$. In Figures 16 and 17, the simulation results for this case are presented. Compared to the agent motion observed in Case 1, we observe here that the desired agent trajectory $\hat{y}$ is smoother and as a result the agent motion is smoother.
H. Discussion on Simulation Results

The motion control law (IV.3) inherently contains two dynamic parts. First is for tracking the target location estimate \( \hat{y}(t) \) generated by the LS based estimation algorithm (IV.2). The second part, \( f(\cdot)\sigma(t) \), adds an excitation signal, whose magnitude diminishes as the target is approached. In the localization algorithm, there are various tuning parameters such as the internal parameters in generation of the covariance matrix \( P(t) \), switching rule of \( A(t) \) in (III.17), initial values of the estimate vectors \( \hat{x}_i \) as well as \( \hat{y}(0) \).

Each of the factors mentioned above affects the agent’s path towards the target. In the simulations, we analyze the effects of the design parameters \( \kappa \), \( h \), different initial agent location \( y(0) \) and target location \( y^* \) values for \( A \) as well as measurement noises and almost collinear sensor settings. Structure of the function \( f(\cdot) \) and initial value of \( \sigma(t) \) have effects on the agent’s path as well. Other norms of the distance error \( (D(t) - D^*) \) can also be employed as the argument of the function \( f \) instead of the infinity-norm. The designer has also the flexibility of choosing \( f(\cdot) \) different than \( f(x) = x \), as long as it satisfies Assumption IV.1.

In addition, we observe the following in all simulations: (i) \( A \) converges to the target location \( y^* \) asymptotically; (ii) the localization algorithm generates the target location estimate \( \hat{y} \) close to its desired value \( y^* \); (iii) distance measurement noise smoothly degrades the performance. These observations are consistent with the boundedness and convergence results established in Section IV. On the other hand, the agent follows a different trajectory in each case for the reasons explained above. The overshoot seen in Case 3 stems from the high \( \kappa \) value which causes sudden motions towards the estimate \( \hat{y}(t) \). The main reason for the circular motion observed in Case 4 is the high \( h \) value which increases the magnitude of the excitation term \( f(\cdot)\sigma(t) \). Smoothness of the agent trajectory in Case 7 follows from the particular value of the initial estimate.
formal robustness analysis and modification of the proposed algorithm for enhancing robustness is left as a future work item. One may consider minimization of other alternative convex cost functions that can uniquely define the target. Convergence properties for these alternatives can also be obtained accordingly.

Results of this paper can easily be applied to real-life scenarios where agent-target distance measurements are not directly obtained by the mobile agent. They may also be combined into multi-agent system control schemes to derive an integrated formation control algorithms. To name a specific example, one may think of the set of sensors as a multiagent system moving in the corresponding space, and merge the autonomous mobile agent to this system with the desired specifications. In this regard, extension of the results obtained in this paper to cover the persistently drifting sensor case is certainly a good path to continue.

**APPENDIX**

**PROOF OF THEOREM IV.1**

(i) Boundedness of the parameter estimates $\hat{e}_i$, $\hat{x}_N$, $\hat{\hat{\alpha}}$, $\hat{\hat{b}}$ and the estimation errors $\tilde{e}_i$, $\tilde{x}_N$, $\tilde{\hat{\alpha}}$, $\tilde{\hat{b}}$ has already been established in Lemma III.1. Boundedness of $\sigma$ and $\dot{\sigma}$ has been established in Lemma IV.2. To see the boundedness of $\dot{\hat{y}}$ and $\ddot{\hat{y}}$, consider the Lyapunov function

$$V_y(t) = \frac{1}{2} \dot{\hat{y}}(t)\dot{\hat{y}}(t).$$

The time derivative of $V_y$ is derived as

$$\dot{V}_y(t) = -\dot{\hat{y}}^T(t)\dot{\hat{y}}(t) + \dot{\hat{y}}^T(t)\dot{\hat{y}}(t) = -\dot{\hat{y}}^T(t)\dot{\hat{y}}(t) + \dot{\hat{y}}^T(t)\dot{\hat{y}}(t).$$

Since, from Lemma III.1 (iv), $\|\tilde{\hat{b}}(t)\| \leq b_{\text{max}}$ for some $b_{\text{max}} > 0$ and for all $t$, (III.17) guarantees that

$$\dot{V}_y \leq -\varepsilon A\dot{\hat{y}}^T(t)\dot{\hat{y}}(t) + b_{\text{max}}\|\dot{\hat{y}}(t)\| = -\varepsilon A\|\dot{\hat{y}}(t)\| - b_{\text{max}}\|\dot{\hat{y}}(t)\|.$$  

(A.11)

(A.11) implies that $\dot{V}_y(t) < 0$ for $\|\dot{\hat{y}}(t)\| > \frac{b_{\text{max}}}{\varepsilon A}$, and hence $\dot{\hat{y}}$ is bounded. This, together with boundedness of $\dot{\hat{y}}$ and $\ddot{\hat{y}}$, implies that $\ddot{\hat{y}}$ is bounded as well.

To establish boundedness of $\dot{\hat{y}}$ and $\ddot{\hat{y}}$, define $e_y(t) \triangleq y(t) - \dot{\hat{y}}(t)$ and rewrite (IV.3) in terms of $e_y$:

$$\dot{e}_y(t) = -\kappa e_y(t) + f (\|D^* - D(t)\|) \dot{\sigma}(t).$$

(A.12)

In (A.12), by Assumption IV.3, the constant $\kappa$ satisfies $\kappa > \sigma > 0$; and by Assumption I.1 and Lemma IV.2, $f (\|D^* - D(t)\|) \dot{\sigma}(t)$ is bounded. Hence, we have $e_y$ and $\dot{e}_y$ are bounded. Since $\dot{\hat{y}}$ and $\ddot{\hat{y}}$ are bounded, this further implies boundedness of $\ddot{\hat{y}}$. Consequently, boundedness of $\dot{\hat{y}}$ follows from (III.4).

(ii) We examine the trajectories of the signals $\tilde{e}_i$, $\tilde{x}_N$ over time, for $t \geq t_{\text{rm}}$, following the steps of the proof of Theorem 4.1 of [8] with minor modifications. Consider the Lyapunov-like function

$$V = \sum_{i=1}^{N-1} \left( \frac{3}{\alpha} \delta_i^2 + \tilde{e}_i^T P^{-1} \tilde{e}_i \right) + \frac{3}{\alpha} \delta_N^2 + \tilde{x}_N^T P^{-1} \tilde{x}_N,$$

(A.13)
with $\gamma_\delta > \frac{\rho_{\min}}{\tilde{A} \delta} > 0$. Using the system equations in Lemma III.1 (ii), the time derivative of $V$ is derived as

$$
\dot{V} = \sum_{i=1}^{N-1} \left( -3\delta_i^2 - \nu_i^2 - 2\nu_i \delta_i \right) - 3\delta_N^2 - \bar{x}_N \left( 2\phi_T - \frac{dP^{-1}}{dt} \right) \bar{x}_N - 2\bar{x}_N \phi \delta_N 
$$

(A.14)

Defining $\nu_i \triangleq \tilde{e}_i^T \phi$ for $i = 1, \ldots, N-1$ and $\nu_N \triangleq \bar{x}_N^T \phi$, and noting that, from (III.6), for $t \geq t_{rm}$,

$$
\frac{dP^{-1}}{dt} = \left\{ -\beta P^{-1} + \phi \phi^T, \text{ if } \lambda_{\max}(P(t)) < \rho_{\max}, \right. \left. 0, \text{ otherwise.} \right\}
$$

(A.15)

(A.14) implies that

$$
\dot{V} \leq \sum_{i=1}^{N-1} \left( -3\delta_i^2 - \nu_i^2 - 2\nu_i \delta_i \right) - 3\delta_N^2 - \nu_N^2 - 2\nu_N \delta_N 
\leq \sum_{i=1}^{N} \left( -\frac{\delta_i^2}{2} - \frac{1}{2} \nu_i^2 \right) \leq 0, \forall t \geq t_{rm};
$$

(A.16)

and further for any $t \geq t_{rm}$ at which $\lambda_{\max}(P(t)) < \rho_{\max}$ and, hence, $\frac{dP^{-1}}{dt} = 0$,

$$
\dot{V} \leq \sum_{i=1}^{N} \left( -\frac{\delta_i^2}{2} - \frac{1}{2} \nu_i^2 \right) - \beta \sum_{i=1}^{N-1} \tilde{e}_i P^{-1} \tilde{e}_i \beta \bar{x}_N P^{-1} \bar{x}_N 
\leq -k_V \dot{V}
$$

(A.17)

for $\delta_V = \min(\alpha/3, \beta)$. Using (A.16),(A.17), from LaSalle-Yoshizawa Theorem, we have that the lumped signal $[\delta_1, \ldots, \delta_N, \tilde{e}_1^T, \ldots, \tilde{e}_{N-1}^T, \bar{x}_N^T, \phi^T]^T$ converges to the set $\Omega = \left\{ [\delta_1, \ldots, \delta_N, \tilde{e}_1^T, \ldots, \tilde{e}_{N-1}^T, \bar{x}_N^T, \phi^T]^T | \dot{V} = \delta_t = \nu_t = 0 \right\}$.

(A.18)

Note that on $\Omega$ we have

$$
\dot{\tilde{e}}_i(t) = \ddot{\tilde{e}}_i(t) = -P(t)\phi(t)\phi^T(t)\tilde{e}_i(t) - P(t)\phi(t)\delta_i(t) = 0,
\dot{\bar{x}}_N(t) = \ddot{\bar{x}}_N(t) = -P(t)\phi(t)\phi^T(t)\bar{x}_N(t) - P(t)\phi(t)\delta_N(t) = 0.
$$

Therefore, $\tilde{e}_i(t) = \tilde{e}_i$ and $\bar{x}_N(t) = \bar{x}_N$, for some constant vectors $\tilde{e}_i$, $\bar{x}_N$. This further implies that $\dot{A}(t) = \tilde{A}$ and $\dot{b}(t) = \tilde{b}$, for a constant matrix $\tilde{A}$ and a constant vector $\tilde{b}$. Furthermore, because of continuity of $\tilde{A}$ and the fact that $\lambda_{\min}(\tilde{A}(t)) \geq \epsilon_A$ for all $t$, $A$ is positive definite as well, and hence

$$
\bar{y}(t) \rightarrow \bar{A}^{-1} \bar{b}, \quad \hat{y}(t) \rightarrow 0
$$

asymptotically as well.

We now show that along the trajectories on $\Omega$, $y(t)$ converges to $y^*$ asymptotically. Taking the time derivative of $\tilde{e}_i^T \phi$ and $\bar{x}_N^T \phi$ on $\Omega$ yields

$$
\frac{d}{dt} (\tilde{e}_i^T \phi(t)) = \tilde{e}_i^T \dot{\phi}(t) = 0 \quad \forall i \in \{1, \ldots, N-1\},
$$

(A.20)

and

$$
\frac{d}{dt} (\bar{x}_N^T \phi(t)) = \bar{x}_N^T \dot{\phi}(t) = 0.
$$

(A.21)

This, together with (III.4), leads to

$$
-\alpha \tilde{e}_i^T \phi(t) + \tilde{e}_i^T \dot{y}(t) = \tilde{e}_i^T \dot{y}(t) = \frac{d}{dt} (\tilde{e}_i^T \phi(t)) = 0 \quad \forall i \in \{1, \ldots, N-1\},
$$

(A.22)

and

$$
-\alpha \bar{x}_N^T \phi(t) + \bar{x}_N^T \dot{y}(t) = \bar{x}_N^T \dot{y}(t) = \frac{d}{dt} (\bar{x}_N^T \phi(t)) = 0
$$

(A.23)

on $\Omega$.

Hence, with (IV.3), (A.19), (A.22), and (A.23), on $\Omega$ there holds:

$$
\tilde{e}_i^T (\hat{y} - \bar{y}) = -\kappa \tilde{e}_i^T (y - \bar{y}) + f(D^*, D) \tilde{e}_i^T \sigma = 0 \quad \forall i \in \{1, \ldots, N-1\},
$$

(A.24)

$$
\bar{x}_N^T (\hat{y} - \bar{y}) = -\kappa \bar{x}_N^T (y - \bar{y}) + f(D^*, D) \bar{x}_N^T \sigma = 0.
$$

(A.25)

(A.24) and (A.25) imply that

$$
f(D^*, D(t)) \tilde{e}_i^T \sigma(t) = f(D^*, D(t)) \bar{x}_N^T \sigma(t) = 0
$$

(A.26)

on $\Omega$. To see that

$$
\tilde{e}_i = \bar{x}_N = 0,
$$

(A.29)

we use contradiction. Assume that (A.29) does not hold. Then, because of Assumption IV.2-(iii), we necessarily have that $f(D^*, D(t)) = 0$ which is equivalent to $y = y^*$. Thus, by the equation $\kappa \tilde{e}_i^T (y - \bar{y})(t) = 0$, there holds $\tilde{e}_i^T = \bar{x}_N = 0$, which contradicts the assumption.

Now, in order to conclude that $y(t)$ asymptotically converges to $y^*$ along the trajectories in $\Omega$, we define the Lyapunov function

$$
L(t) = \frac{1}{2} \tilde{e}_y^T(t) \tilde{e}_y(t),
$$

(A.30)

where $\tilde{e}_y$ is defined in (IV.9). Then,

$$
\dot{L} = -2\kappa L + f(D^*, D(t)) \tilde{e}_y \tilde{\sigma}
\leq -2\kappa L + f(D^*, D(t)) ||\tilde{e}_y|| ||\tilde{\sigma}||
\leq -2\kappa L + ||\tilde{e}_y|| ||\tilde{\sigma}||
= -2(\kappa - ||\tilde{\sigma}||) L,
$$

(A.31)

(A.32)

(A.33)

(A.34)

where we have applied (IV.10). Under Assumption IV.3 we have that $\dot{L}(t) \leq 0$. Since the largest invariant set satisfying (A.34) in $\Omega$ is constituted by $y(t) = y^*$, from LaSalle’s Invariance Principle, we conclude that $\tilde{e}_y(t)$ converges to 0 and hence, $y(t)$ converges to $y^*$ asymptotically.

REFERENCES


