Guaranteed maximum power point tracking by scalar iterations with quadratic convergence rate

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Abstract— **In this paper the problem of maximum power point tracking (MPPT) is considered. We show that the problem has a unique solution and it can be reduced to the problem of finding the unique root of a single variable scalar function. We show that Newton's iterations can be applied to the problem of finding this root quadratically fast for an initialisation that is independent of the parameters of the MPPT problem. The results of applying the approach to 1,000,000 randomly-generated instances of the MPPT problem are presented and consistency with the analysis is observed.**

I. Introduction

One of the problems that has attracted much attention in the areas of power electronics and renewable energy generation is the problem of maximum power point tracking. The output current and the parameters of a photovoltaic (PV) array varies with the insolation level, G, and ambient temperature, T. Like any other engineering practice it is desired to maximize the power output of PV arrays for any solar and temperature conditions. In principle, this is equivalent to finding the operating terminal voltage for the PV array that maximizes the power output. This point is called *Maximum Power Point* (MPP) and the act of ensuring that the PV array always operate at MPP is termed *Maximum Power Point Tracking* (MPPT). It has been observed – through experimentations going back to 1950's, e.g. see [1]–[3] – that the MPP is unique under even illumination. At MMP the rate of change of the power with respect to the voltage is equal to zero. In this paper, we establish that the MPP is unique and show that any properly tuned steepest ascent algorithm converges to the MPP.

There is a plethora of algorithms for finding the MPP. Many of the existing algorithms are based on heuristics – with no convergence guarantees – and mainly rely on perturbing the voltage (or current), measuring the power output, and comparing the observed output power with the previous values in order to maximize the power output. Then based on this observation the direction of the next step of perturbation is determined, i.e. if the output power is increased the next point is chosen along the same direction, otherwise the opposite direction is explored. For example, perturb and observe and incremental conductance are two of such methods. For a comprehensive review of such methods the reader may refer to [4]–[8] and the references there-in. However, no formal convergence analysis is provided for the cases where these methods are applied to the MPPT problem. In fact, it is common knowledge that the iterations might exhibit oscillatory behaviours [4]–[6]. These methods in principle treat the MPPT problem as a derivative free optimisation problem – for a review of derivative free problems see [9], [10, Chapter 9] and [11]. As a result, one suspects that even if a convergence results is provided for these methods the rate of the convergence is sublinear, see [11]. Other methods based on numerically calculating the first and the second order derivatives of the power with respect of the terminal voltage has been also studied that give rise to linear convergence rates in a neighbourhood of the optimal solution [12], [13]. However, those methods rely on the existence of the solution, which have not been proved previously. Further, they can only prove local convergence to the optimum. It is hoped that the uniqueness result presented in this paper sheds some light on why many of the existing methods manage to find the MPP, albeit slowly and with no convergence guarantees[1].

In this paper, the MPPT problem is revisited and it is established that there is indeed a unique global maximum at which the Karush-Kuhn-Tucker (KKT) conditions are satisfied analytically. Second, it is shown that solving the MPPT problem is equivalent to finding the unique root of a scalar function. Third, the Newton's iterations are applied to find this root and it is established that one can always initialise the Newton's iterations at a point that is independent of the parameters of the problem such that the root of the scalar function is found quadratically fast. Surprisingly, this initial point is shown to be always 1. Fourth, we propose a simple practical approach for estimating the parameters of the MPPT that change with environmental variables, i.e. those parameters that change with solar irradiation and temperature. This is in contrast with methods that numerically calculate derivatives, e.g., [4]–[6], [12], [13], and allow the presented method to have a superior convergence rate.

The idea of applying steepest ascent algorithms or Newton iterations to solve the MPPT problem is not original to this paper, e.g. see [14], [15]. However, in this paper we rigorously establish that the Newton's iterations solves the MPPT problem quadratically fast and can be implemented efficiently.

The outline of the paper is as follows. First, we introduce the required preliminaries and define the maximum power point tracking problem. In Section III, we cast the problem finding the maximum power point as an optimsiation

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[1] In [8] the results rely on the uniqueness of the MPP and this is assumed to be true from experimental observations.

problem and show that it has a unique solution. In Section IV, the Newton's iterations are introduced and it is shown that they can be employed to solve the MPPT problem. In Section V we propose a circuit along with an estimation approach for computing the parameters of the problem that are influenced by the environmental variables, i.e. those parameters that change due to changes in the level of solar irradiation and/or temperature. In Section VI, we extensively apply the proposed Newton iterations to solve one million randomly-generated instances of the MPPT problem. Concluding remarks and future research directions are discussed in Section VII.

II. Preliminaries and Problem Formulation

A solar panel is composed of parallel and series cells and if these cells are well-matched and evenly illuminated then the panel can be modelled as the circuit of Fig. 1 [8]. The source I_L corresponds to the current generated due to solar irradiation where often it is given by $I_L = n_p I_{Ph}$ with n_p being the number of parallel cells in the panel and I_{Ph} the current generated at each of these cells and is a function of the solar irradiation, G. Furthermore, R_p models the overall losses due to leakage currents in the panel, R_s captures the actual electric resistance of the panel. Additionally, $I_D = I_s\left(e^{\alpha(V+IR_s)} - 1\right)$ where I_s is the reverse saturation current of the diode, $\alpha = q/(n_s F K_B T)$ with $q = 1.6022 \times 10^{-19}\,\mathrm{C}$ being equal to the electron charge, $F \in (0, 1]$ is the ideality factor of the diode, $K_B = 1.3807 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ is the Boltzmann's constant, and T is the absolute temperature. The reader may consult [4]–[8] for more information regarding the modelling of PV panels. Thus, I and V satisfy the following equation:

$$\ell(I, V) = I - I_L + I_s\left(e^{\alpha(V+IR_s)} - 1\right) + \frac{V + IR_s}{R_p} = 0. \tag{1}$$

Now we are interested in finding the upper bounds of I and V. We start by finding the upper bound for I. It can be seen from the monotonicity of $\ell(V, I)$ in I (strictly increasing) and the facts that for any given V, $\lim_{I\to\infty} \ell(V, I) = +\infty$ and $\lim_{I\to-\infty} \ell(V, I) = -\infty$ there is a unique I that satisfies (1) for each V. In the light of the continuous differentiability of $\ell(I, V)$ defined in (1) as well as the fact that $\partial\ell/\partial I \neq 0$, and from the implicit function theorem there exists a unique and continuously differentiable function $f(\cdot)$ such that $I = f(V)$. Thus, it is possible to comment on the variation of I with respect to V. From elementary calculus and positiveness of R_s, R_p, I_s, and α, we have:

$$\frac{df}{dV} = -\frac{\alpha I_s e^{\alpha(V+IR_s)} + 1/R_p}{1 + \alpha I_s R_s e^{\alpha(V+IR_s)} + R_s/R_p} < 0.$$

Similarly (or alternatively by invoking the inverse function theorem), there exists a unique and continuously differentiable function $g(\cdot)$ such that $V = g(I)$ $(f^{-1}(\cdot) = g(\cdot))$ where

$$\frac{dg}{dI} = -\frac{1 + \alpha I_s R_s e^{\alpha(V+IR_s)} + R_s/R_p}{\alpha I_s e^{\alpha(V+IR_s)} + 1/R_p} < 0.$$

Therefore, I is strictly decreasing in V, and V is strictly decreasing in I. Consequently, with the requirement that $I \geq 0$ and $V \geq 0$ (so that the system generates power rather than consuming it), the upper bound on I, denoted by I_{sc}, is its value when $V = 0$, i.e. the solution to the following equation:

$$I - I_L + I_s\left(e^{\alpha I R_s} - 1\right) + \frac{IR_s}{R_p} = 0, \tag{2}$$

Similarly, the upper bound on V, termed V_{oc}, is obtained at $I = 0$ and uniquely satisfies the following equation:

$$-I_L + I_s\left(e^{\alpha V} - 1\right) + \frac{V}{R_p} = 0. \tag{3}$$

Remark 1: The solutions to (2) and (3) are the same as the short-circuit current and the open-circuit voltage, respectively.

Before defining the MPPT problem we comment on the the values for parameters R_s, R_p, I_s, α, and I_L. The main parameters that are influenced by environmental factors are α and I_L, where solar irradiation impacts I_L and temperature influences the value of α. For the majority of this paper we assume that the constant *structural parameters of the system*, i.e. R_s, R_p, and I_s, are known *a priori* and I_L and α are accessible through radiation and temperature measurements. Later in the paper, we propose a method for estimating I_L and α using a simple circuitry that facilitates measuring V_{oc} and I_{sc}.

The MPPT problem can be stated as the following:

Problem 1: The MPPT problem is the problem of finding non-negative $V^\star$ and $I^\star$ that maximize $P = VI$ while satisfy (1) for given parameters R_s, R_p, I_s, α, and I_L.

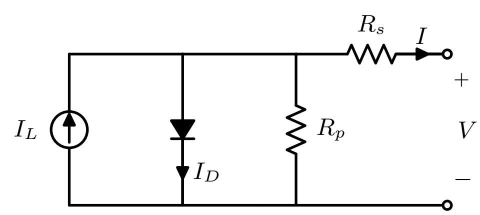

Fig. 1. The equivalent circuit of a PV panel.

III. Uniqueness of MPP

The goal is to find the solution to the following optimisation problem

$$\begin{aligned} \max \quad & VI \\ \text{subject to} \quad & I - I_L + I_s\left(e^{\alpha(V+IR_s)} - 1\right) + \frac{V + IR_s}{R_p} = 0 \\ & V \geq 0, \quad I \geq 0, \end{aligned} \tag{4}$$

where I_L, I_s, R_p, R_s, and α are initially assumed to be known positive real constants. In what follows we study the properties of (4). First, we recast the problem in a form that is more amenable to the type of algebraic arguments that we pursue in the rest of this paper. To this aim, let

$$z \triangleq e^{\alpha(V+IR_s)}. \tag{5}$$

Problem (4) can be rewritten as

$$\begin{aligned} \max \quad & I\frac{\log z}{\alpha} - I^2 R_s \\ \text{subject to} \quad & I - I_L + I_s(z-1) + \frac{\log z}{\alpha R_p} = 0 \\ & z = e^{\alpha(V+IR_s)}, \quad V \geq 0, \quad I \geq 0. \end{aligned} \tag{6}$$

We have the following result for (6).

Proposition 1: The optimisation problem (6) has a unique maximum at $[V^\star, I^\star, z^\star]$ where $z^\star$ is the unique solution of the following equation

$$\log z = \frac{(I_s(z-1) - I_L)\left(1 + 2R_s\left(\alpha I_s z + \frac{1}{R_p}\right)\right)}{-\frac{1}{\alpha R_p} - \left(\frac{1}{\alpha} + \frac{2R_s}{\alpha R_p}\right)\left(\alpha I_s z + \frac{1}{R_p}\right)}, \tag{7}$$

and

$$I^\star = I_L - I_s(z^\star - 1) - \frac{\log z^\star}{\alpha R_p}, \tag{8}$$

$$V^\star = \frac{\log z^\star}{\alpha} - I^\star R_s. \tag{9}$$

Proof: First, we note that from (2) and (3) the set defined by the constraints of (6) is defined by:

$$\begin{aligned} & I - I_L + I_s\left(e^{\alpha(V+IR_s)} - 1\right) + \frac{V + IR_s}{R_p} = 0 \\ & 0 \leq V \leq V_{oc}, \quad 0 \leq I \leq I_{sc}, \\ & \min(e^{\alpha V_{oc}}, e^{\alpha R_s I_{sc}}) \leq z \leq \max(e^{\alpha V_{oc}}, e^{\alpha R_s I_{sc}}). \end{aligned} \tag{10}$$

This set is closed and bounded (compact) and the cost function and all the constraints are continuously differentiable over the set. Moreover, $[0, I_{sc}, e^{\alpha R_s I_{sc}}]^\top$ and $[V_{oc}, 0, e^{\alpha V_{oc}}]^\top$ are the only points where the inequality constraints in (6) are active. Furthermore, through basic algebra it can be see that the value of the cost function at these points is the same and equal to zero, while the value of the cost function is strictly positive when neither of the inequality constraints is active. Hence, the cost function attains its maximum at a point that both inequality constraints are inactive.

Define the corresponding Lagrangian function of (6) as

$$\begin{aligned} \mathcal{L}(V, I, z, \lambda_1, \lambda_2, \nu_1, \nu_2) = & I\frac{\log z}{\alpha} - I^2 R_s \\ & - \lambda_1\left(I - I_L + I_s(z-1) + \frac{\log z}{\alpha R_p}\right) \\ & - \lambda_2\left(z - e^{\alpha(V+IR_s)}\right) \\ & - \nu_1 I - \nu_2 V. \end{aligned} \tag{11}$$

Since, none of the inequality constraints is active (i.e. $I > 0$ and $V > 0$), $\nu_1 = \nu_2 = 0$. Writing the first order necessary KKT conditions for optimality [10] yields

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{I}{\alpha z} - \lambda_1\left(I_s + \frac{1}{\alpha R_p z}\right) - \lambda_2 = 0 \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial I} = \frac{\log z}{\alpha} - 2IR_s - \lambda_1 + \lambda_2 \alpha R_s e^{\alpha(V+IR_s)} = 0 \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial V} = \lambda_2 \alpha R_s e^{\alpha(V+IR_s)} = 0 \tag{14}$$

$$I - I_L + I_s(z-1) + \frac{\log z}{\alpha R_p} = 0 \tag{15}$$

$$z - e^{\alpha(V+IR_s)} = 0. \tag{16}$$

First, observe that from (14) that $\lambda_2 = 0$. Hence, the condition can be rewritten as

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{I}{\alpha z} - \lambda_1\left(I_s + \frac{1}{\alpha R_p z}\right) = 0 \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial I} = \frac{\log z}{\alpha} - 2IR_s - \lambda_1 = 0 \tag{18}$$

$$I - I_L + I_s(z-1) + \frac{\log z}{\alpha R_p} = 0 \tag{19}$$

$$z - e^{\alpha(V+IR_s)} = 0. \tag{20}$$

Thus,

$$\lambda_1 = \frac{\log z}{\alpha} - 2R_s\left(I_L - I_s(z-1) - \frac{\log z}{\alpha R_p}\right).$$

Consequently,

$$\begin{aligned} & \left(I_L - I_s(z-1) - \frac{\log z}{\alpha R_p}\right) \\ & - \left(\frac{\log z}{\alpha} - 2R_s\left(I_L - I_s(z-1) - \frac{\log z}{\alpha R_p}\right)\right) \\ & \times \left(\alpha I_s z + \frac{1}{R_p}\right) = 0, \end{aligned}$$

which implies that

$$\begin{aligned} & \left(-\frac{1}{\alpha R_p} - \left(\frac{1}{\alpha} + \frac{2R_s}{\alpha R_p}\right)\left(\alpha I_s z + \frac{1}{R_p}\right)\right)\log z \\ & = (I_s(z-1) - I_L)\left(1 + 2R_s\left(\alpha I_s z + \frac{1}{R_p}\right)\right). \end{aligned}$$

Therefore,

$$\log z = \frac{(I_s(z-1) - I_L)\left(1 + 2R_s\left(\alpha I_s z + \frac{1}{R_p}\right)\right)}{-\frac{1}{\alpha R_p} - \left(\frac{1}{\alpha} + \frac{2R_s}{\alpha R_p}\right)\left(\alpha I_s z + \frac{1}{R_p}\right)}. \tag{21}$$

Due to Lemma 1 in the appendix, (21) has a unique solution denoted by $z^\star$. This in turn results in a unique $I^\star$ and consequently a unique $V^\star$ given by (8) and (9). So far we have shown that there is only one point that satisfies the necessary optimality conditions. Next, we show that this point is in the interior of the feasible set and consequently the only solution of the optimisation problem. To this aim, we check if the obtained $I^\star$ and $V^\star$ are positive (i.e. they satisfy the inequality constraints). We show this for $I^\star$ in what follows. From (7) and (8):

$$
\begin{aligned}
I^\star &= I_L - I_s(z^\star - 1) \\
&\quad - \frac{(I_s(z^\star-1)-I_L)\left(1+2R_s\left(\alpha I_s z^\star + \frac{1}{R_p}\right)\right)}{-1-(R_p+2R_s)\left(\alpha I_s z^\star + \frac{1}{R_p}\right)} \\
&= \frac{(\alpha R_p I_s z + 1)\left(I_L + I_s(1-z)\right)}{1+(R_p+2R_s)\left(\alpha I_s z^\star + \frac{1}{R_p}\right)}
\end{aligned}
$$

From Lemma 1 we know $z^\star \in (1, 1+\frac{I_L}{I_s})$. Thus, $I^\star > 0$. Similarly, one can show that $V^\star > 0$. This concludes the proof. ■

Proposition 1 establishes that in fact any ascent algorithm, e.g. gradient ascent algorithm, with properly selected parameters, e.g. step-sizes, can solve (6) exactly. Since, problem (6) is equivalent to problem (4), the MPPT problem, one concludes that the same class of ascent algorithms with properly selected parameters solves the MPPT problem. In the next section, we use Proposition 1 to propose an efficient algorithm that solves the MPPT problem.

IV. Newton Iterations for MPPT

There is a rich family of numerical methods for solving multivariable optimisation problems of the type described by (6). Gradient ascent and Newton methods have been applied for solving the MPPT problem [14], [15]. However, no guarantees for convergence or bounds on the rate of convergence have been reported so far.

In what follows, we take another route to solving the MPPT problem; instead of solving (6) or (4), we use the result of Proposition 1 and solve the scalar equation (7). To this aim, we use Newton's iterations to achieve quadratic convergence and show that one can always find an initial guess for the Newton's method that solves (7). Additionally, it is demonstrated that the value of this initial guess is independent of the parameters of the problem.

Let h be a twice differentiable scalar function, the Newton iterations for finding the roots of h are described by

$$
z_{k+1} = z_k - \frac{h(z_k)}{h'(z_k)}, \tag{22}
$$

where h' is the first derivative of h. For a generic h, Newton's iterations converge quadratically fast to the solution when the initial guess is sufficiently close to the solution. The following theorem is a well-known result, also known as *Fourier Conditions*, for global convergence of the Newton's method [16].

Theorem 1: Let h'' be continuous on an interval $[\underline{z}, \overline{z}]$, containing a root $z^\star$ of h, $h' \neq 0$, and either $h'' \leq 0$ or $h'' \geq 0$ on $[\underline{z}, \overline{z}]$. Then Newtons iterations converge monotonically to $z^\star$ from any point $z_0 \in [\underline{z}, \overline{z}]$ such that $h(z_0)h''(z_0) \geq 0$.

We have the following result when Newton's iterations of the type described by (22) are used to solve (7).

Proposition 2: Consider the iterations (22) for

$$
\begin{aligned}
h(z) &= -\log z \\
&\quad + \frac{(I_s(z-1)-I_L)\left(1+2R_s\left(\alpha I_s z + \frac{1}{R_p}\right)\right)}{-\frac{1}{\alpha R_p} - \left(\frac{1}{\alpha} + \frac{2R_s}{\alpha R_p}\right)\left(\alpha I_s z + \frac{1}{R_p}\right)},
\end{aligned} \tag{23}
$$

and let $z_0 = 1$. Then the sequence $\{z_k\}$ defined by (22) monotonically converges to $z^\star$ with a quadratic rate and

$$
|z_{k+1} - z^\star| \leq \frac{M_1 M_2}{2} |z_k - z^\star|^2,
$$

where $M_1 = h''(1)$ and $M_2 = |1/h'(z^\star)|$.

Remark 2: Due to space constraints, the proofs are removed.

Remark 3: Proposition 2 establishes that Newton iterations can always be initialized such that they converge to $z^\star$ with a quadratic rate.

Remark 4: Consider the convergence factor obtained from Proposition 2. The value of M_1 can be calculated before solving the problem, while the value of M_2 can only be calculated after finding $z^\star$. However, one can calculate an upper bound for M_2. Denote this upper bound M_3 and it is the magnitude of $h'(z)$ as $z \to \infty$ (since $h'' > 0$) and it can be seen that

$$
M_3 = \frac{2R_s R_p \alpha I_s}{R_p + 2R_s}.
$$

V. Estimating I_L and α

As discussed in Section II, the values of I_L and α are directly influenced by environmental variables, particularly solar irradiance and temperature. In this section, we comment on how these two variables can be estimated without deploying specialized radiation and temperature sensors.

Recall (1) and the fact that structural parameters R_s, R_p, and I_s are known. Any pair of measured current and voltage, I and V, yield the following exponential equation in α

$$
I - I_L + I_s\left(e^{\alpha(V+IR_s)} - 1\right) + \frac{V + IR_s}{R_p} = 0, \tag{24}
$$

where I_L and α are unknown. Let two measurements pairs, (I_1, V_1) and (I_2, V_2), be available. By subtracting (24) for the pair of voltage-current measurements, one obtains:

$$
\begin{aligned}
&I_s e^{\alpha(V_1+I_1R_s)} - I_s e^{\alpha(V_2+I_2R_s)} \\
&\quad + I_1 - I_2 + \frac{V_1 - V_2 + (I_1 - I_2)R_s}{R_p} = 0.
\end{aligned} \tag{25}
$$

Due to the monotonicity of $e^{\alpha(V_i+I_iR_s)}$, $i = 1, 2$, in α, (25) has a unique solution if and only if $V_1 + I_1R_s \neq V_2 + I_2R_s$.

Two candidate pairs of current and voltage measurements that we will use in the rest of this section are $(I_{sc}, 0)$ and $(0, V_{oc})$. Using the short-circuit and open-circuit values yields

$$
I_s e^{I_{sc}R_s\alpha} - I_s e^{V_{oc}\alpha} + I_{sc} + \frac{-V_{oc} + I_{sc}R_s}{R_p} = 0. \tag{26}
$$

As stated above (26) has a unique solution if and only if $I_{sc}R_s \neq V_{oc}$. Denote the solution (26) by $\hat{\alpha}$; then the corresponding $\hat{I}_L$ for the PV panel is obtained from:

$$\hat{I}_L = I_s \left(e^{\hat{\alpha} V_{oc}} - 1\right) + \frac{V_{oc}}{R_p}. \tag{27}$$

Hence, having access to values of I_{sc} and V_{oc} makes computation of $\hat{\alpha}$ and $\hat{I}_L$ (the estimated values of α and I_L) possible. Before outlining how we propose to measure I_{sc} and V_{oc} we make the following observation regarding the actual implementation of an MPPT. The output of a solar panel is often connected to a power electronics device such as a DC-AC inverter or a DC-DC converter. In this paper, we assume that device connected to the panel is a DC-DC converter with duty cycle D and conversation ratio $M(D)$. Moreover, let it be connected to a nominal load equal to R_l. It is straightforward to see that at the terminals of the PV panel this load is seen as $R_g = R_l/M(D)^2$. Thus, to extract maximum power from the panel R_g should be equal to $R^\star = V^\star/I^\star$ and consequently, the duty cycle of the converter should be selected in a way that $M(D) = \sqrt{R_l/R^\star}$.

Assume that the solar panel is connected to a buck-boost converter (any other type of DC-DC converter can be used as well.). Similar to [7], [8] we propose a circuit of the form depicted in Fig. 2. Assume that the converter is at steady state. The on- and off- cycles of transistors Q_M and Q_1 are depicted in Fig. 3. The short-circuit current, I_{sc}, is measured during the time that Q_M is conducting and Q_1 is not conducting. Similarly, the open-circuit voltage is measured when both Q_M and Q_1 are off.

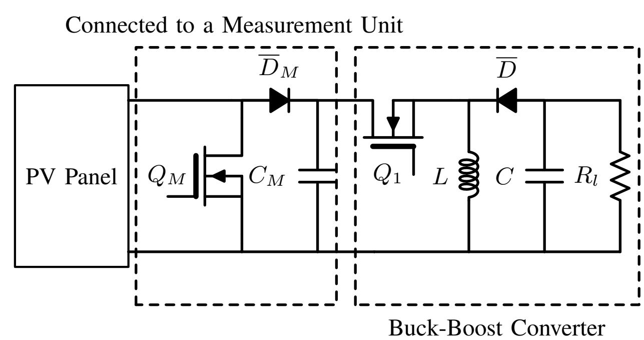

Fig. 2. The PV panel connected to a buck-boost converter via a circuitry for measuring I_{sc} and V_{oc}. Note that the choice of the DC-DC converter has no impact on the preceding circuit that connects to the panel.

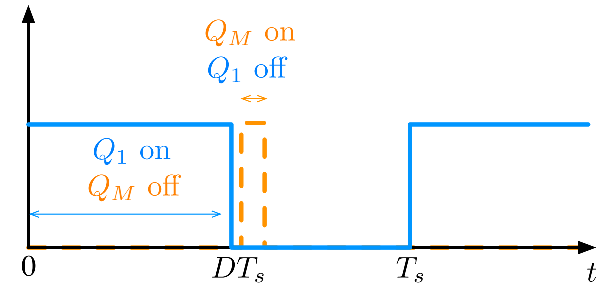

Fig. 3. The on- and off- cycles of transistors Q_1 and Q_M where D_1 is the duty cycle of Q_1 and D_M is the duty cycle of Q_M. The switching frequency of both is assume to be equal to $1/T_s$.

Parameter	Value Range
R_s	$[0.1\,\Omega, 5\,\Omega]$
R_p	$[1 \times 10^5\,\Omega, 1 \times 10^8\,\Omega]$
I_L	$[1\,\text{A}, 10\,\text{A}]$
I_s	$5 \times 10^{-10}\,\text{A}$
T	$[265\,\text{K}, 310\,\text{K}]$
K_B	$1.3807 \times 10^{-23}\,\text{J}\,\text{K}^{-1}$
q	$1.6022 \times 10^{-19}\,\text{C}$
n_s	$[40, 50]$
F	$[0.9, 1]$

TABLE I

THE RANGE OF PARAMETERS OF THE MPPT PROBLEMS SOLVED VIA (22). FOR EACH SCENARIO THE PARAMETERS WERE SELECTED UNIFORMLY FROM GIVEN RANGES. THE VALUE OF I_s IS CONSISTENT WITH THE REVERSE SATURATION CURRENT OF A SILICON DIODE. OTHER PARAMETER RANGES ARE CHOSEN SUCH THAT THEY INCLUDE THE SPECIFICATIONS OF AN ASSORTMENT OF COMMERCIAL PV PANELS: (REC PEAK ENERGY SERIES, SUNTECH (STP255 - 20/WD, STP250 - 20/WD, STP245 - 20/WD), SUNPOWER E20/435 SOLAR PANEL, AND BP SOLAR BP 4175.)

VI. Numerical Example

In this section, we study the performance of the Newton's iterations described by (22) with h as in (7), to solve the MPPT problem through an extensive set of numerical examples. To this aim 1,000,000 randomly-generated instances of the MPPT problem with parameters selected according to Table I were solved and the average and the worst-case (maximum) relative distance from $z^\star$, i.e. $|z_k - z^\star|/z^\star$, versus the iterations number is depicted in Fig. 4.

In line with Proposition 2, the initial guess for each problem is chosen to be $z_0 = 1$ From Fig. 4 in can be easily observed that the relative error (both in average and worst-case) drops below 10^{-8} on average after 31 iterations. Moreover, the absolute error drops below 10^{-6} after the same number of steps. The solution time for each problem is 1.6×10^{-4} s on a 2.8 GHz Intel Core i7 MAC laptop using Python 3.4. We conclude this section by commenting on the relationships between possible error in the calculated $z^\star$ and the computed $I^\star$ and $V^\star$. The first order relationships between the inaccuracies in the optimal values $I^\star$ and $V^\star$ denoted by $\Delta I^\star$ and $\Delta V^\star$, respectively, and the error in computing $z^\star$, $\Delta z^\star$, are:

$$\Delta I^\star = I_s \Delta z^\star - \frac{1}{\alpha R_p} \frac{\Delta z^\star}{z^\star},$$

$$\Delta V^\star = \frac{1}{\alpha} \frac{\Delta z^\star}{z^\star}.$$

Thus, after 31 steps the maximum error in the computed value of $I^\star$ and $V^\star$ in the worst-case are of orders 10^{-16} and 10^{-12}, respectively.

VII. Concluding Remarks and Future Directions

In this paper we revisited the MPPT problem and demonstrated that it has a unique solution. Furthermore, we established that solving the MPPT problem is equivalent to

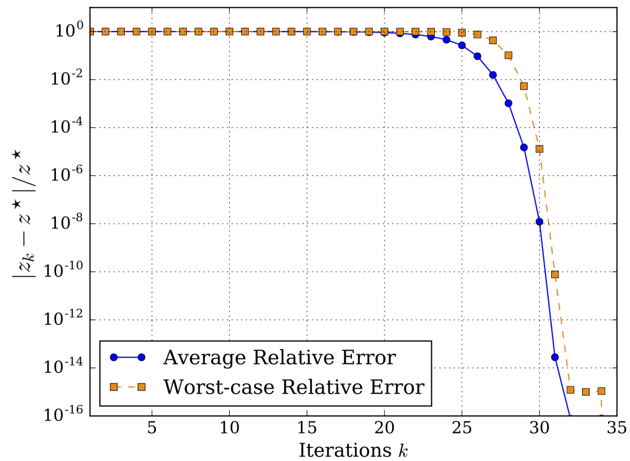

Fig. 4. The error versus iteration number for 1,000,000 instances of the MPPT problem with parameters described in Table I.

finding the unique root of a scalar single variable function and proved that the Newton's iterations find this root and consequently solve the MPPT problem quadratically fast. We note that all the existing derivative-free algorithms for solving the MPPT problem exhibit sublinear convergence. Hardware implementation of the proposed MPPT constitutes one of the possible future directions. Additionally, higher order methods, e.g. Halley or Traub iterations [16], [17], for solving the MPPT problem can be considered.

Appendix

Lemma 1: Consider the equation

$$h(z) = 0, \tag{28}$$

where

$$h(z) = -\log z + r(z), \tag{29}$$

$$r(z) = \frac{(I_s(z-1) - I_L)\left(1 + 2R_s\left(\alpha I_s z + \frac{1}{R_p}\right)\right)}{-\frac{1}{\alpha R_p} - \left(\frac{1}{\alpha} + \frac{2R_s}{\alpha R_p}\right)\left(\alpha I_s z + \frac{1}{R_p}\right)} \tag{30}$$

$$= -d\frac{(z-a)(z+b)}{(z+c)}, \tag{31}$$

and

$$a = 1 + \frac{I_L}{I_s}, \tag{32}$$

$$b = \frac{1}{2R_s \alpha I_s} + \frac{1}{R_p \alpha I_s}, \tag{33}$$

$$c = \frac{1}{(R_p + 2R_s)\alpha I_s} + \frac{1}{R_p \alpha I_s}, \tag{34}$$

$$d = \frac{2R_s R_p \alpha I_s}{R_p + 2R_s}. \tag{35}$$

The following statements are true.

1) Equation (28) has a unique solution.
2) Let the solution to (28) be denoted by $z^\star$, then
$$1 < z^\star < 1 + \frac{I_L}{I_s}.$$

Lemma 2: Consider $h(z)$ as described in (29). For $z \in (0, +\infty)$, the following statements are true:

1) $h'(z) < 0$;
2) $h''(z) > 0$;
3) $h'''(z) < 0$.

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