Early Termination of NMPC Interior Point Solvers: Relating the Duality Gap to Stability

Andrei Pavlov, Iman Shames and Chris Manzie

***Abstract*— In this paper, we present an early termination condition for the primal-dual interior-point method for application in nonlinear model predictive control (MPC) problems. The condition verifies the prescribed suboptimality level of a feasible iteration of the algorithm and enables one to employ the feasible suboptimal solution without jeopardising the stability of the system. The distinguishing property of the proposed condition is its primal-dual formulation, which allows the proposed early termination of the algorithm to be independent from any values computed on the previous time instances. Numeral experiments on a nonlinear planar multirotor system and a comparison of the proposed early termination condition with the existing methods are provided.**

I. INTRODUCTION

Model predictive control (MPC) is a feedback control method based on numerical optimisation. A well known advantage of MPC is that it can utilise dynamic system models in optimising the prescribed performance metric whilst satisfying inherent constraints of the system. However, for some situations the complexity of the optimisation problem itself precludes the solution in real time.

In the case of convex formulation of MPC problem one can compute the optimal control law explicitly (Explicit MPC [1], [2]) or approximate it to sufficient degree (Approximate Explicit MPC [3], [4]) to avoid solving the optimisation problem in real-time. However, these methods are known to have scalability issues associated with the nature of parametric problems. Thus, these approaches are usually limited to relatively low dimensional systems. Nevertheless, while sparse grids can help to mitigate the curse of dimensionality in approximating the control law for the dynamical systems of moderate dimensions [5], [6], high degree of freedom systems are still out of the scope of the explicit methods.

Therefore, extensive attention has been paid to suboptimal solutions of the optimisation problems to reduce the computational costs associated with MPC [7]–[9]. The basic idea of proposed approaches is to verify sufficient reduction of a Lyapunov function candidate for a given feasible control sequence. It allows early termination of the optimisation algorithm thereby reducing the complexity. Motivated by similar reasons, several techniques of input parametrisation, aimed at reducing the number of decision variables, have also been considered [10]–[12].

To further reduce the computational burden associated with MPC efficient algorithms for solving optimisation problems should be used [13]. For example, primal-dual interior-point methods are a special class of algorithms for numerical optimisation that run in polynomial time and are very efficient in practice. In [14] it was shown that the structure of the problem could be exploited to reduce the complexity of primal-dual algorithm iterations. There exist similar opportunities to exploit the suboptimality criteria required for a control sequence to preserve stability.

The remainder of this paper is organized as follows. Section 2 gives a gives a brief introduction to stability results in MPC framework and introduces the idea of primal-dual interior-point method to solve the optimisation problem. In Section 3 the main contribution of this paper is discussed, where a primal-dual formulation of the suboptimality-based early termination condition is proposed. In Section 4 we provide numerical comparisons of existing and proposed approaches for a nonlinear planar multirotor system and illustrate advantages of the proposed approach. Conclusions are summarized in Section 5.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Model predictive control

Consider a discrete-time system of the following form:

$$x^+ = f(x, u), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a sufficiently smooth map, which for a given current state $x \in \mathbb{R}^n$ and a control input $u \in \mathbb{R}^m$ assigns the state $x^+ \in X$ at the next sampling instant. Moreover, the origin of the system is an equilibrium, i.e., $f(0, 0) = 0$.

Assume that the states and inputs of the system are subject to the following constraints:

$$x \in \mathcal{X},\ u \in \mathcal{U}, \tag{2}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are closed convex polyhedrons whose interior contains the origin.

Define the stage cost $q(x, u)$ for being in state x and taking action u, where

$$\begin{aligned} &q(x, u) \text{ is sufficiently smooth,}\\ &q(x, u) \text{ is convex in } x \text{ and } u,\\ &q(x, u) > 0 \text{ for } (x, u) \neq (0, 0),\\ &q(0, 0) = 0. \end{aligned} \tag{3}$$

We introduce an auxiliary function $p(x)$, which is strictly convex in x, equals zero at the origin and strictly positive everywhere else, and a compact set $\mathcal{X}_f \subseteq \mathcal{X}$, which contains

Andrei Pavlov, Iman Shames and Chris Manzie are with Department of Electrical and Electronic Engineering, The University of Melbourne, Victoria 3010, Australia. apavlov@student.unimelb.edu.au, iman.shames@unimelb.edu.au, manziec@unimelb.edu.au

the origin. The function and the set must satisfy the following property:

$$\begin{aligned}&\forall x \in \mathcal{X}_f \;\exists u \in \mathcal{U} :\\ &f(x,u) \in \mathcal{X}_f \text{ and } p\big(f(x,u)\big) + q(x,u) \le p(x).\end{aligned} \tag{4}$$

Consider the task of driving the system to the origin from the current state x, while minimising a sum of stage costs over the prediction horizon N, where N is a positive integer, and satisfying the constraints at all times. We formulate a finite time constrained optimal control problem to be solved at each time step as follows:

$$\begin{aligned} J^\star(x) = \min_{\boldsymbol{\xi},\boldsymbol{u}} \quad & J(\boldsymbol{\xi},\boldsymbol{u}) = \sum_{i=0}^{N-1} q(\xi_i,u_i) + p(\xi_N)\\ s.t. \quad & h(x,\boldsymbol{\xi},\boldsymbol{u}) = 0\\ & g(\boldsymbol{\xi},\boldsymbol{u}) \le 0 \end{aligned} \tag{5}$$

where $\boldsymbol{u} = [u_0,\dots,u_{N-1}]$ and $\boldsymbol{\xi} = [\xi_0,\dots,\xi_N]$ are vectors of decision variables, $h(x,\boldsymbol{\xi},\boldsymbol{u}) = 0$ incorporates constraints on the initial condition and the dynamics, i.e. $\xi_0 = x$ and $\xi_{i+1} = f(\xi_i,u_i)$ for $i = 0,\dots,N-1$, and $g(\boldsymbol{\xi},\boldsymbol{u}) \le 0$ accounts for $\xi_i \in \mathcal{X}$, $u_i \in \mathcal{U}$ and $\xi_N \in \mathcal{X}_f$. In further sections we refer to this problem as the MPC problem.

The equality constraints in (5) can be substituted in the objective function, so it becomes a function of the current state and control inputs: $J(x,\boldsymbol{u})$. We denote a global optimiser of (5) as $\boldsymbol{u}^\star = [u_0^\star,\dots,u_{N-1}^\star]$ and use notation $\boldsymbol{u}^\star(x)$ to point out that it is a function of x. From here the key idea of the MPC framework is easy to see: at each sampling instance the optimisation problem, described by (5), is solved for a current state x to form the feedback law $u_0^\star(x)$.

Asymptotic stability of the origin of the system under MPC control can be guaranteed by the existence of a strictly decreasing Lyapunov function. Under the aforementioned assumptions, the function $J^\star(x) = J(x,\boldsymbol{u}^\star(x))$ is a suitable Lyapunov function candidate.

Remark 1. *In general it is hard to find a globally optimal solution for the non-convex optimisation problem. In practice local optimality of the solution is enough for MPC control [15], however additional care should be taken to avoid switching between different locally optimal trajectories.*

B. Stability of suboptimal MPC

As it was outlined previously, the idea of MPC is to use the first element of the optimiser of a finite time constrained optimal control problem as a control input. However, in the case of limited computational resources or fast sampling, exact computation of the local optimiser at every sampling instant might be prohibitive due to the time constraints. As a typical optimisation algorithm approaches a solution iteratively, it makes an intermediate iteration a good candidate candidate for control purposes.

In what follows we present a slightly modified version of [16, Theorem 14.1], which introduces sufficient conditions for the stability under a feasible sub-optimal control law. It was modified to include the nonlinear dynamics of the system and account for the whole stage cost.

Theorem 1. *Consider a nonlinear system* (1) *subject to constraints* (2)*, a corresponding MPC problem* (5)*, such that* (3) *and* (4) *hold, and a level of suboptimality* $\gamma \in (0,1)$*. If the control law* $\tilde{\boldsymbol{u}}(x) = [\tilde{u}_0(x),\dots,\tilde{u}_{N-1}(x)]$*, formed as a feasible solution to* (5) *for a given* x*, satisfies the following for all* $x \in \mathcal{X}_0 \subseteq \mathcal{X}$*:*

$$J\big(x,\tilde{\boldsymbol{u}}(x)\big) \le J^\star(x) + \gamma q\big(x,\tilde{u}_0(x)\big),$$

where $J^\star(x)$ *is the optimal cost, then the origin of the system is asymptotically stable under the* γ*-suboptimal control law* $\tilde{u}_0(x)$ *with a domain of attraction* $\mathcal{X}_0$*.*

Proof. The proof follows the same steps as the proof of Theorem 14.1 in [16] and terminates at the equation (14.7). □

Remark 2. *The result of Theorem 1 enables one to employ a feasible suboptimal solution of* (5) *without jeopardising the stability of the system. However, to be able to use the condition one needs to know the optimal cost at the current state.*

C. Primal-dual interior-point method applied to MPC

Interior-point methods are a class of algorithms that can be used to find a local optimiser of a nonlinear optimisation problem of a sufficient degree of regularity. Here we consider a primal-dual interior-point method for solving finite time constrained optimal control problems arising in the MPC framework.

For the optimisation problem, described by (5), we define the Lagrangian function $L(x,\boldsymbol{\xi},\boldsymbol{u},\lambda,s)$ as follows:

$$L(x,\boldsymbol{\xi},\boldsymbol{u},\lambda,s) = J(\boldsymbol{\xi},\boldsymbol{u}) + \lambda^T h(x,\boldsymbol{\xi},\boldsymbol{u}) + s^T g(\boldsymbol{\xi},\boldsymbol{u}),$$

where λ and s are dual variables, or KKT multipliers, for the equality and inequality constraint respectively.

Here we assume that the problem satisfies the linear independence constraint qualification (LICQ) at the optimal point (this can be replaced by a weaker condition), thus KKT conditions are the first-order necessary conditions for a solution of (5) to be locally optimal:

$$\begin{aligned} \nabla_{(\boldsymbol{\xi},\boldsymbol{u})} L(x,\boldsymbol{\xi},\boldsymbol{u},\lambda,s) &= 0,\\ h(x,\boldsymbol{\xi},\boldsymbol{u}) &= 0,\\ g(\boldsymbol{\xi},\boldsymbol{u}) + y &= 0,\\ (s,y) \ge 0,\; s_j y_j = 0,\; j &= 1,\dots,l; \end{aligned} \tag{6}$$

where we introduced a vector of slack variables $y \in \mathbb{R}^l$ (l is a number of inequality constraints).

The idea of the primal-dual interior-point method is to replace the complementarity slackness condition $s_j y_j = 0$ with a perturbed one $s_j y_j = \mu$ for $j = 1,\dots,l$, where $\mu > 0$ is a barrier parameter. Let k be the iteration counter. Now we iteratively approach a solution of (6) with a sequence $(\boldsymbol{\xi}^{(k)},\boldsymbol{u}^{(k)},\lambda^{(k)},s^{(k)},y^{(k)})$ by taking damped steps in a Newton direction, obtained from the linearisation of (6) at the current iteration. At every step additional care should be taken to keep $(s^{(k)},y^{(k)})$ at a sufficient distance from the

Data: System of KKT conditions and a starting point
Result: Solution with a specified level of accuracy
Perform the initialisation;
while *termination condition* Π *is not satisfied* **do**
  Adjust the barrier parameter;
  Compute the search direction;
  Perform the line search;
  if *the trial point is accepted* **then**
    Take the step;
  else
    Perform the feasibility restoration procedure;
  end
end

Algorithm 1: A high-level representation of the primal-dual interior-point method

non-negativity boundary, while forcing $\mu^{(k)} \xrightarrow{k\to\infty} 0$ until some termination condition Π is satisfied.

In general, the feasibility of the primal-dual iterations with respect to the vector-valued equality constraints of (6) is not automatically guaranteed. If the iterations are infeasible the search direction has to be modified, so that some measure of infeasibility is minimised. In the case of quadratic or linear programs, once the step results in primal and dual feasible variables, all subsequent iterates remain feasible.

As the solution of (6) is obtained as a limit point of the sequence of the primal-dual iterations, a finite termination procedure is required. This requires a termination condition Π to be defined. Usually, the rule checks whether the accuracy level achieved on the current iteration is within the user provided error tolerance level or not. However, one can use Theorem 1 as a starting point for proposing a termination condition that results in a stabilising control input. However, it must be noted that Theorem 1 cannot be readily used due to the fact that the exact value of the optimal cost is unknown.

This idea has been explored in [7] and the proposed solution was to construct a bounding sequence, based the values of the objective functions from the previous time instants. The early termination condition relies on enforcing a sufficient decrease in the cost function at the current time instant compared to the value from the previous time instant:

$$J(x, \boldsymbol{u}^{(k)}) \leq J(\bar{x}, \bar{\boldsymbol{u}}) - \alpha q(\bar{x}, \bar{u}_0), \tag{7}$$

where $\alpha \in (0, 1)$, x is the current state, $\bar{x}$ and $\bar{\boldsymbol{u}}$ are the state and control respectively from the previous time instant. Contrary, the termination condition proposed in this paper is independent of values computed in the previous time instants.

D. Problem formulation

As stated above, the aim of this paper is to propose a termination condition Π for the primal-dual interior-point method for MPC, that results in a stable closed-loop system. The problem of interest is given below.

Problem 1. *Given the MPC problem and Algorithm 1, propose a termination condition* Π *such that the feedback law obtained from the algorithm results in a stable closed-loop system. Furthermore, it is required that the termination condition to be independent of the optimal solution or solutions obtained at the previous time instances.*

III. Main result

In this section we state the main contribution of this paper – the solution to the Problem 1, i.e. a termination condition Π for the primal-dual interior-point method for MPC, that results in a stable closed-loop system.

Theorem 2. *Consider a nonlinear system* (1) *subject to constraints* (2)*, a corresponding MPC problem* (5)*, such that* (3) *and* (4) *hold, and a level of suboptimality* $\gamma \in (0, 1)$*. Denote by* k *the iteration counter and by* $(\boldsymbol{\xi}^{(k)}, \boldsymbol{u}^{(k)}, \lambda^{(k)}, s^{(k)}, y^{(k)})$ *the* k*-th iteration of the primal-dual interior-point algorithm, and denote by* $u_0^{(k)}$ *the first element of* $\boldsymbol{u}^{(k)}$.

Let the algorithm terminate at a feasible k*-th iteration where the condition*

$${s^{(k)}}^T y^{(k)} \leq \gamma q(x, u_0^{(k)}) \tag{8}$$

is satisfied. If $u_0^{(k)}$ *is used to form the MPC control law then the closed-loop system is stable.*

Proof. Consider the Lagrange function associated with (5):

$$L(x, \boldsymbol{\xi}, \boldsymbol{u}, \lambda, s) = J(\boldsymbol{\xi}, \boldsymbol{u}) + \lambda^T h(x, \boldsymbol{\xi}, \boldsymbol{u}) + s^T g(\boldsymbol{\xi}, \boldsymbol{u}),$$

where λ and s are dual variables for the equality and inequality constraint respectively. For the nonlinear optimisation problem the weak duality holds and can be written as follows:

$$\max_{s\geq 0, \lambda} \min_{\boldsymbol{\xi}, \boldsymbol{u}} L(x, \boldsymbol{\xi}, \boldsymbol{u}, \lambda, s) \leq \min_{\boldsymbol{\xi}, \boldsymbol{u}} \max_{s\geq 0, \lambda} L(x, \boldsymbol{\xi}, \boldsymbol{u}, \lambda, s). \tag{9}$$

Now let's consider the k-th iteration of the primal-dual interior point method which is feasible, i.e. the iteration satisfies perturbed KKT conditions:

$$\begin{aligned} \nabla_{(\boldsymbol{\xi}, \boldsymbol{u})} L(x, \boldsymbol{\xi}^{(k)}, \boldsymbol{u}^{(k)}, \lambda^{(k)}, s^{(k)}) &= 0, \\ h(x, \boldsymbol{\xi}^{(k)}, \boldsymbol{u}^{(k)}) &= 0, \\ g(\boldsymbol{\xi}^{(k)}, \boldsymbol{u}^{(k)}) + y^{(k)} &= 0, \\ (s^{(k)}, y^{(k)}) &> 0. \end{aligned} \tag{10}$$

For the given iteration the stationary condition with respect to $\boldsymbol{\xi}$ and $\boldsymbol{u}$ is achieved, as it follows from the first equation of (10). The conditions for a local maximum with respect to λ and s are, however, not fulfilled. Thus,

$$L(x, \boldsymbol{\xi}^{(k)}, \boldsymbol{u}^{(k)}, \lambda^{(k)}, s^{(k)}) \leq \max_{s\geq 0, \lambda} \min_{\boldsymbol{\xi}, \boldsymbol{u}} L(x, \boldsymbol{\xi}, \boldsymbol{u}, \lambda, s). \tag{11}$$

By considering (9) and (11) we can conclude that:

$$L(x, \boldsymbol{\xi}^{(k)}, \boldsymbol{u}^{(k)}, \lambda^{(k)}, s^{(k)}) \leq \min_{\boldsymbol{\xi}, \boldsymbol{u}} \max_{s\geq 0, \lambda} L(x, \boldsymbol{\xi}, \boldsymbol{u}, \lambda, s),$$

which along with the fact that the iteration is feasible, yields:

$$J(x, \boldsymbol{u}^{(k)}) - {s^{(k)}}^T y^{(k)} \leq J^\star(x).$$

Hence, if

$${s^{(k)}}^T y^{(k)} \leq \gamma q(x, u_0^{(k)}),$$

then

$$J\big(x, \boldsymbol{u}^{(k)}\big) \leq J^\star(x) + \gamma q(x, u_0^{(k)}).$$

Thus, the requirements of Theorem 1 are satisfied and we conclude the stability of the closed loop system. □

Remark 3. *The proposed early termination condition is very cheap to evaluate and requires neither the knowledge of the value of the objective function from the previous time instance, nor the knowledge of the optimal cost for the current state.*

Remark 4. *In comparison to the early termination condition, described by* (7)*, our criterion, as it will be demonstrated in the numerical experiments section, results in a desirable solution in fewer steps of the primal-dual interior-point method. It is conjectured that this is due to the fact that the proposed termination condition does not rely on the sufficient decrease of the cost function compared to the previous time instants.*

IV. Numerical experiments

In this section we compare the average number and variance of iterations taken by the primal-dual algorithm (IPOPT [17]) before one of the following termination conditions is satisfied: exact solution withing user specified error tolerance (10^{-8}), existing condition (7) on a sufficient decrease of the cost function for $\alpha = 0$, and the proposed condition (8) for $\gamma = 1$. The early termination conditions apply only to feasible iterations (primal and dual infeasibility less than 10^{-3}).

Here we consider the dynamics obtained by discretising the quadrotor continuous-time dynamics by Euler method with a time step h of 0.2 seconds, while restricting its spatial motion to $y - z$ plane (2D quadrotor). The system has the following nonlinear dynamics affected by bounded additive disturbance d (note that no disturbances $d = 0$ were assumed in MPC problem formulation, thus $d = 0$ will be used in the control synthesis):

$$x^+ = x + f_c(x, u)h + d,$$

where $f_c(x, u)$ is given by

$$f_c(x,u) = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1/m \sin(x_3) & 0 \\ 1/m \cos(x_3) & 0 \\ 0 & 1/M \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The state vector x consist of y-z-θ coordinates and velocities along the corresponding axes. The mass m, inertia M and standard gravity g are assumed to be 0.5, 1 and 9.8 respectively.

We choose a quadratic stage cost $q(x,u) = x^T Q x + (u - u_{eq})^T R (u - u_{eq})$, where $Q = \frac{1}{2} I_6$ and $R = I_2$ (I_6 and I_2 are the unity matrices of appropriate dimension), u_{eq} is a steady state control input at the equilibrium. The control inputs are subject to constraints $u_0 \in [0, 10]$ and $u_1 \in [-2, 2]$. Auxiliary function $p(x)$ is the LQR cost-to-go function, computed for the linearised dynamics around $x = 0$ and $u = u_{eq}$. Set $\mathcal{X}_f$ is a box of unit size around the origin, i.e. $\mathcal{X}_f = \{x \mid \|x\|_\infty \leq 0.5\}$. This choice of $\mathcal{X}_f$ was made to be a subset of the control invariant set computed for the linearised dynamics under the LQR control law, while the control invariant and stabilising properties for the nonlinear dynamics were verified numerically (details on more systematic approach [18]).

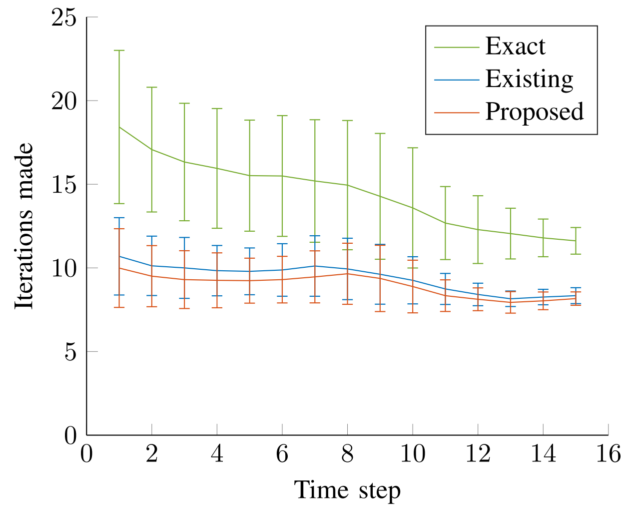

Fig. 1: The number of iterations before termination versus time-steps without warm-starting. The lines represent the mean and the bars represent the standard deviation.

In the first set of simulations we randomly initialise the system inside the box $\{x \mid \|x\|_\infty \leq 1\}$, while the disturbances are considered to act only in the direction of steepest ascent of the optimal cost function:

$$d = 5 \cdot 10^{-3} \frac{\nabla J^\star(x)}{\|\nabla J^\star(x)\|}.$$

These disturbances should prevent early termination of the existing method, and so (in some sense) represent a more challenging scenario for the algorithm.

Based on Fig. 1 we conclude that if warm-starting is not used, the algorithm terminates almost equally soon for both of the early termination conditions with a marginal advantage of the proposed approach. We next consider the same disturbance scenario, but warm start the algorithm to represent more likely implementations. Warm starting is performed with both primal and dual variables from the previous iteration shifted by a time step, where the missing primal values are generated by the system dynamics under LQR control and missing dual variables are the sensitivity of the LQR cost-to-go function (for equality constraints) and zeros (for the inequality constraints).

As it can be seen from Fig. 2 the proposed approach requires fewer iterations in average to find a control input of a prescribed quality and clearly has smaller variance in comparison with the existing early termination condition. The algorithm makes more iterations to force a sufficient

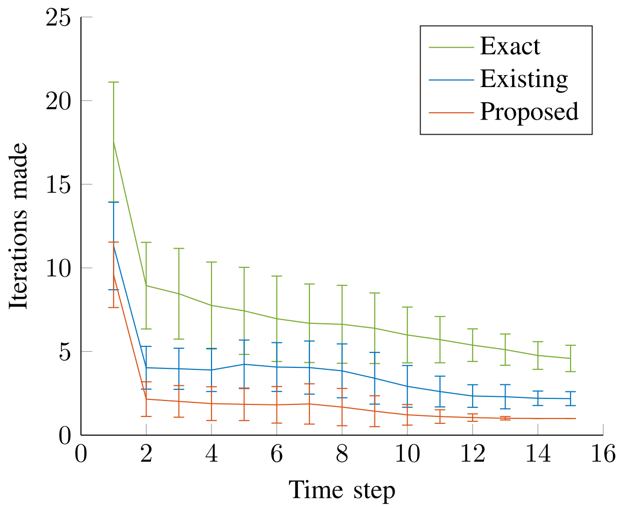

Fig. 2: The number of iterations before termination versus time-steps with warm-starting. The lines represent the mean and the bars represent the standard deviation.

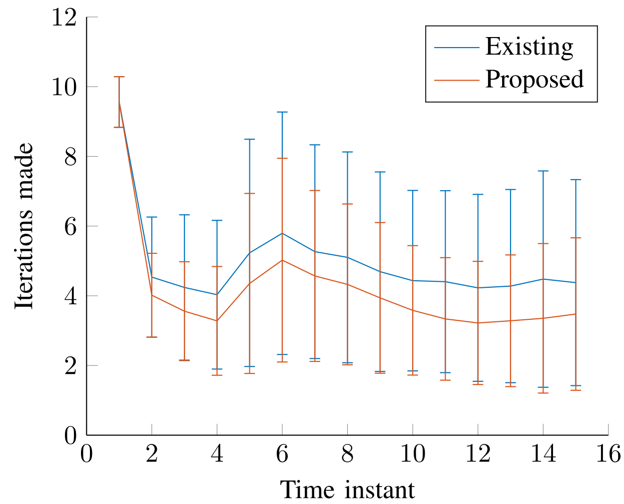

Fig. 3: The number of iterations before termination versus time-steps with warm-starting for bounded uniformly distributed disturbances. The lines represent the mean and the bars represent the standard deviation.

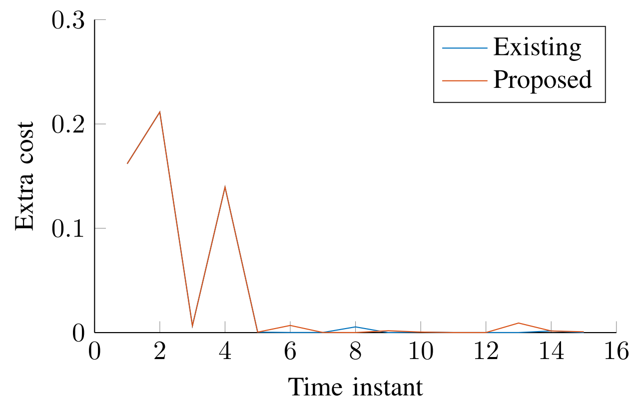

Fig. 4: Extra cost (compared to the optimal cost) along the system trajectory for two early termination criteria.

reduction in the cost function, than it necessary to just verify the level of suboptimality of the current iterate.

In actuality, these disturbance scenarios will be unrealistically pessimistic in that they always act against the existing early termination condition. To provide a more realistic disturbance scenario we consider uniform random disturbance trajectories for the warm started algorithms, specifically,

$$\|d\|_\infty \leq 0.2.$$

Here we randomly initialise the system inside the box of size 0.4 centred at $x = [2; 2; \pi/4; 0; 0; 0]^T$ and investigate the number of primal-dual iterations made before satisfying the early termination conditions when at the half of cases the disturbances are driving the system in the descent direction of the cost function.

As it can be concluded from Fig. 3, the proposed approach requires $\approx 18\%$ less computational resources in average to find a control input of a prescribed quality. Here the peak computational requirements are decreased by $\approx 14\%$ when the proposed condition is used.

Next we keep random disturbances as before and consider a single scenario, where the system was initialised in some random state. We compare the extra cost (defined as a difference between sub-optimal cost and optimal cost at the current state) along the closed-loop trajectories, obtained by using one of two considered early termination criteria. As present on Fig. 4, both early termination criteria provide solutions of similar quality.

V. Conclusion

Primal-dual interior-point method is an important tool in finding a locally optimal solution of an nonlinear optimisation problem. As the solution is obtained as a limit point of iterations, the number of iterations made by the algorithm highly depends on the finite termination procedure utilised. When a model predictive control is implemented on a low-powered embedded system, it might be desirable to terminate the algorithm earlier without compromising the stability of the closed-loop system.

We proposed an early termination condition, which verifies the prescribed suboptimality level of a feasible solution without the knowledge of the optimal solution itself. Moreover, while the existing condition forces a sufficient reduction of the cost function compared to the previous time instant, the proposed early termination condition is independent from solutions obtained at the previous time instances. Based on the numerical experiments we conclude that the condition has a clear advantage if the algorithm is warm-started from the solution computed at the previous time instant. In comparison with the existing method, utilisation of the proposed condition reduces the average number of iteration made by the algorithm, as well as its variability. There is, however, only a marginal advantage of the proposed condition if warm-starting is not performed or external disturbances are insignificant. While this analysis applies for the specific case studies, the proposed approach may be important for certain embedded controller applications.

REFERENCES

[1] Petter TøNdel, Tor Arne Johansen, and Alberto Bemporad. An algorithm for multi-parametric quadratic programming and explicit mpc solutions. *Automatica*, 39(3):489–497, 2003.

[2] Arun Gupta, Sharad Bhartiya, and PSV Nataraj. A novel approach to multiparametric quadratic programming. *Automatica*, 47(9):2112–2117, 2011.

[3] F Scibilia, Sorin Olaru, and M Hovd. Approximate explicit linear mpc via delaunay tessellation. In *Control Conference (ECC), 2009 European*, pages 2833–2838. IEEE, 2009.

[4] Colin N Jones and Manfred Morari. Polytopic approximation of explicit model predictive controllers. *IEEE Transactions on Automatic Control*, 55(11):2542–2553, 2010.

[5] Sean Summers, Colin N Jones, John Lygeros, and Manfred Morari. A multiresolution approximation method for fast explicit model predictive control. *IEEE Transactions on Automatic Control*, 56(11):2530–2541, 2011.

[6] Wei Kang and Lucas C Wilcox. Mitigating the curse of dimensionality: sparse grid characteristics method for optimal feedback control and hjb equations. *Computational Optimization and Applications*, 68(2):289–315, 2017.

[7] Pierre OM Scokaert, David Q Mayne, and James B Rawlings. Suboptimal model predictive control (feasibility implies stability). *IEEE Transactions on Automatic Control*, 44(3):648–654, 1999.

[8] Darryl DeHaan and Martin Guay. A real-time framework for model-predictive control of continuous-time nonlinear systems. *IEEE Transactions on Automatic Control*, 52(11):2047–2057, 2007.

[9] Melanie Nicole Zeilinger, Colin Neil Jones, and Manfred Morari. Real-time suboptimal model predictive control using a combination of explicit mpc and online optimization. *IEEE Transactions on Automatic Control*, 56(7):1524–1534, 2011.

[10] Raphael Cagienard, Pascal Grieder, Eric C Kerrigan, and Manfred Morari. Move blocking strategies in receding horizon control. *Journal of Process Control*, 17(6):563–570, 2007.

[11] Rohan C Shekhar and Chris Manzie. Optimal move blocking strategies for model predictive control. *Automatica*, 61:27–34, 2015.

[12] Ravi Gondhalekar and Jun-ichi Imura. Least-restrictive move-blocking model predictive control. *Automatica*, 46(7):1234–1240, 2010.

[13] Christopher V Rao, Stephen J Wright, and James B Rawlings. Application of interior-point methods to model predictive control. *Journal of optimization theory and applications*, 99(3):723–757, 1998.

[14] Alexander Domahidi, Aldo U Zgraggen, Melanie N Zeilinger, Manfred Morari, and Colin N Jones. Efficient interior point methods for multistage problems arising in receding horizon control. In *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, pages 668–674. IEEE, 2012.

[15] W-H Chen, Donald J Ballance, and John O'Reilly. Model predictive control of nonlinear systems: Computational burden and stability. *IEE Proceedings-Control Theory and Applications*, 147(4):387–394, 2000.

[16] Francesco Borrelli, Alberto Bemporad, and Manfred Morari. *Predictive control for linear and hybrid systems*. Cambridge University Press, 2017.

[17] Andreas Wächter and Lorenz T Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical programming*, 106(1):25–57, 2006.

[18] Hong Chen and Frank Allgöwer. A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 34(10):1205–1217, 1998.