Minimax strategy in approximate model predictive control

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Abstract

It is known that a model predictive control law for a linear dynamical system with a linear or quadratic cost function can be explicitly computed as a piece-wise affine function. However, the number of regions required grows rapidly with the horizon length, the number of states and constraints limiting the deployment of explicit solutions to relatively small MPC problems, and motivating approximate solutions requiring less storage for online implementation. Unfortunately, the offline computation required to generate the approximate solution can be very high using many existing algorithms. In this paper, we propose a new procedure to generate the approximate solution based on barycentric interpolation whilst retaining a (less conservative) certification of the controller. This novel certification procedure requires solving a number of small-scale multiparametric linear programs together with convex optimization problems. During the online implementation of the approximate MPC, only a small linear program has to be solved to evaluate the control law. The efficacy of the proposed approach is demonstrated through both simulation and also in application to a canonical cart-pole stabilization problem.

Key words: Model Predictive Control; Receding-Horizon control; Minimax problem; Multiscale Methods; Explicit MPC; Fast MPC.

1 Introduction

The well known advantage of Model Predictive Control (MPC) is that it can utilize dynamic system models in optimizing the cost whilst satisfying inherent constraints of the system. However, for some situations the complexity of the optimization problem itself precludes the solution in real time. Therefore, extensive attention has been paid to sub-optimal solutions of the optimization problems to reduce the computational costs associated with MPC [1–3]. Motivated by similar reasons, several techniques of input parametrization, aimed at reducing the number of decision variables, have also been considered [4–6].

One may try to compute the optimal control law explicitly (Explicit MPC) as a solution of corresponding multiparametric problem and use it without solving the MPC problem online [7]. However, this approach is limited to linearly constrained quadratic problems and is known to have scalability issues associated with the nature of algorithms for solving these problems [7–9]. Thus, the ideas of Semi-Explicit [10] or Approximate Explicit MPC [11] might be utilized to parametrize or approximate the optimal control law.

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During the offline phase of Approximate Explicit MPC, a sub-optimal control law is defined over the some region of state space. As an important requirement, the control law has to produce sufficient decrease of a Lyapunov function candidate over the region where it is defined to guarantee asymptotic stability of the origin. Once this fact is established, the region is said to have a *stability certificate*, otherwise the region should be partitioned (with further refinement of the control law) until the stability certificate is obtained.

One of the challenges in this approach is that the certification procedure becomes computationally intractable for an arbitrary defined control law. A proposed approach to deal with this issue is the use of barycentric coordinates [11]. In such approaches, the control law is obtained by a special convex combination of *a priori*-computed optimal control inputs at finitely many points and stability certificate is tested by means of convex programming.

In the online phase, the partition and the sub-optimal control law are stored and used to compute the control input at demand. To evaluate the sub-optimal control law, first, one solves a *point location problem*: for a given system state the region, which contains this state, has to be found among all stored regions. Then, the control input is computed based on the sub-optimal control law defined for this region.

With these factors in mind, several approaches have been presented in the literature to approach the problem of partitioning: triangulation [12], Hausdorff distance optimal polytopic partition [13] and orthogonal multiresolution partition [14]. The aforementioned ways of implementing an approximate MPC solution are subject to trade-off between computational efforts, memory requirements for storing the partition, and a level of conservatism of a certification procedure used.

The main contribution of this paper is a novel way of computing barycentric coordinates and a corresponding stability certificate, which greatly reduces the number of regions in the final partition in comparison to the state-of-the-art approach. Proposed barycentric coordinates are computed as a solution to a small-scale linear program, which makes them computationally tractable. The connection between the certificate and the coordinates reduces the level of conservatism in the certification procedure. We also introduce an anisotropic way of the partitioning, which uses sensitivity of the underlying optimization problem.

The remainder of this paper is organized as follows. Section 2 gives a brief introduction to stability and recursive feasibility results in Approximate Explicit MPC framework. In Section 3 ,the first contribution of this paper is discussed, where a less conservative stability certification procedure is proposed together with a barycentric function corresponding to this certificate. In Section 4, a method to decrease the total number of regions by performing anisotropic partitioning is introduced. In Section 5, we provide numerical comparisons of existing and proposed approaches and illustrate applicability of the proposed approach on a two-wheeled balancing robot. Concluding remarks are presented in Section 6.

2 Problem formulation and theoretical background

The main problem of interest is presented below.

Problem 1 *Reduce the number of regions in Approximate Explicit MPC formulations by:*

(1) Proposing a certification procedure and a barycentric function that result in stability preserving control laws under less conservative requirements than existing methods;
(2) Extending an orthogonal partitioning strategy to establish the certification whilst requiring fewer regions than existing methods.

In this section, the relevant preliminaries leading up to the approximate MPC problem are introduced, which allows consideration of alternative certification and partitioning strategies.

2.1 Model predictive control

Consider a linear controllable discrete-time system with state and input matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$:

$$x(t+1) = Ax(t) + Bu(t). \tag{1}$$

Vectors $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are system state and input respectively, subject to the following constraints:

$$x(t) \in \mathcal{X},\ u(t) \in \mathcal{U},\ \forall t \in \mathbb{Z}^+, \tag{2}$$

where $\mathbb{Z}^+$ is the set of non-negative integers, $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are closed convex polyhedrons whose interior contains the origin. Define function $q(x,u)$ to be a cost for being in state x and taking action u:

$$\begin{aligned} &q(x,u) \text{ is convex in } x \text{ and strictly convex in } u,\\ &q(x,u) > 0 \text{ everywhere, except } q(0,0) = 0. \end{aligned} \tag{3}$$

We also introduce a control invariant set $\mathcal{X}_f \subseteq \mathcal{X}$ for the system and a strictly convex function $p(x)$: $p(0) = 0$ and

$$\forall x \in \mathcal{X}_f,\ \exists u \in \mathcal{U} :\ p(Ax + Bu) - p(x) \le -q(x,u). \tag{4}$$

Consider the task of driving the system to the origin from the current state $x := x(t)$, while minimizing a cost over the prediction horizon N and satisfying the constraints:

$$\begin{aligned} J^\star(x) = \min_{\boldsymbol{\xi},\boldsymbol{u}} \quad & J(\boldsymbol{\xi},\boldsymbol{u}) = \sum_{k=0}^{N-1} q(\xi_k,u_k) + p(\xi_N)\\ s.t. \quad & k = 0,\dots,N-1:\\ & \xi_{k+1} = A\xi_k + Bu_k,\\ & \xi_k \in \mathcal{X},\ u_k \in \mathcal{U},\\ & \xi_N \in \mathcal{X}_f,\ \xi_0 = x, \end{aligned} \tag{5}$$

where $\boldsymbol{u} = [u_0,\dots,u_{N-1}]$ and $\boldsymbol{\xi} = [\xi_0,\dots,\xi_N]$ are vectors of decision variables.

Equality constraints in (5) can be eliminated, so objective becomes a function of the current state and control inputs: $J(x,\boldsymbol{u})$. We denote the minimizer of (5) as $\boldsymbol{u}^\star = [u_0^\star,\dots,u_{N-1}^\star]$ and use notation $\boldsymbol{u}^\star(x)$ to point out that it is a function of x. The key idea of the MPC framework is to use $u(t) = u_0^\star(x(t))$ as a feedback control law.

We denote with $\mathcal{X}_0 \subseteq \mathcal{X}$ the set of all admissible states, i.e., (5) is feasible for all $x \in \mathcal{X}_0$. If feasibility of (5) for $x(0)$ implies its further feasibility along the closed-loop trajectory $x(t)$ for $t > 0$, then MPC control is called *recursively feasible*. Asymptotic stability of the origin of the system under the recursively feasible MPC control can be guaranteed by the existence of a strictly decreasing Lyapunov function. If $J^\star(x)$ satisfies (4), then it is a suitable Lyapunov function candidate [15].

2.2 Approximate Explicit MPC

In what follows we present a slightly modified version of [11, Theorem 14.1], which introduces sufficient conditions for the stability under a feasible sub-optimal control law $\tilde{u}_0(x)$, i.e., the law obtained from a sequence of first elements of feasible solutions $\tilde{\boldsymbol{u}}(x)$ of (5).

Theorem 2 *Consider a linear system subject to constraints, as described by (1) and (2), and the control law $\tilde{u}_0(x)$ obtained from a sequence of first elements of feasible solutions $\tilde{\boldsymbol{u}}(x)$ of (5). If $\tilde{\boldsymbol{u}}(x)$ for all $x \in \mathcal{X}_0$ satisfies*

$$J(x, \tilde{\boldsymbol{u}}(x)) \leq J^\star(x) + \gamma(x),$$

where $\gamma(x) < q(x, \tilde{u}_0(x))$ and $J^\star(x)$ is the optimal cost, then the origin of the system is asymptotically stable under suboptimal control law $\tilde{u}_0(x)$ with a domain of attraction $\mathcal{X}_0$.

PROOF. The proof follows the same steps as the proof of Theorem 14.1 in [11] and terminates at the equation (14.8). □

Remark 3 *As it will become clear later, multiple convenient choices of $\gamma(x)$ are possible, e.g., $\gamma(x) = \gamma_0 q(x, 0)$, where $\gamma_0 \in (0, 1)$, or $\gamma(x) = \gamma_0 q(x, \tilde{u}_0(x)) - \epsilon$, where $\gamma_0 = (0, 1]$ and $\epsilon > 0$.*

2.2.1 Barycentric interpolation

It is intractable to use Theorem 2 to provide a stability certificate for an arbitrary feasible control law. Thus, we consider a framework of generalized barycentric coordinates [1] and use it to define a feasible control law accompanied by a tractable stability and suboptimality certification procedure. Consider a set of points $v^i \in \mathbb{R}^n$ for $i \in \mathcal{I} = \{1, \ldots, d\}$ and denote with V its convex hull:

$$V := \big\{ \sum_{i \in \mathcal{I}} \alpha_i v^i \mid \alpha_i \geq 0,\ \sum_{i \in \mathcal{I}} \alpha_i = 1 \big\}.$$

For any point inside V there is a (possibly non-unique) vector of coordinates, as defined in what follows:

Definition 4 (Barycentric coordinates) *Consider a set of points $v^i \in \mathbb{R}^n$ for $i \in \mathcal{I} = \{1, \ldots, d\}$, its convex hull V, and a point $x \in V$. We call $\Lambda(x)$ the set of barycentric coordinates for point x:*

$$\Lambda(x) := \big\{ \lambda \in \mathbb{R}^d \mid \lambda_i \geq 0,\ \sum_{i \in \mathcal{I}} \lambda_i = 1,\ \sum_{i \in \mathcal{I}} \lambda_i v^i = x \big\}.$$

[1] The word "generalized" is omitted for the sake of brevity in the remainder of the paper.

We call a mapping between the elements of V and its barycentric coordinates a *barycentric function.*

Definition 5 (Barycentric function) *Consider a set of points $v^i \in \mathbb{R}^n$ for $i \in \mathcal{I} = \{1, \ldots, d\}$ and its convex hull V. A mapping $w : \mathbb{R}^n \to \mathbb{R}^d$ is called a barycentric function if for every $x \in V$ its image $w(x)$ is its barycentric coordinates, i.e.,*

$$w_i(x) \geq 0,\ \sum_{i \in \mathcal{I}} w_i(x) = 1,\ \sum_{i \in \mathcal{I}} w_i(x) v^i = x.$$

The introduction of these functions enable the way of defining a sub-optimal control sequence

$$\tilde{\boldsymbol{u}}(x) = \sum_{i \in \mathcal{I}} w_i(x) \boldsymbol{u}^\star(v^i), \tag{6}$$

where $\boldsymbol{u}^\star(v^i)$ are the optimizers of (5) for $v^i \in \mathcal{X}_0$ and $w(x)$ is a barycentric function. Since (5) is feasible for all v^i, then the convex hull property is satisfied for $\tilde{\boldsymbol{u}}(x)$, i.e., the convex combination of points from a convex region is a point inside the region. Thus, $\tilde{\boldsymbol{u}}(x)$ is a feasible control sequence which leads to state constraints satisfaction [14]. Moreover, for all x inside V the following holds (Lemma 14.2 in [11]):

$$J(x, \tilde{\boldsymbol{u}}(x)) \leq \sum_{i \in \mathcal{I}} w_i(x) J^\star(v^i) \leq \max_{\lambda \in \Lambda(x)} \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i). \tag{7}$$

Corollary 6 *If $\delta_V \geq 0$ (or $\delta_V \geq \epsilon > 0$ if $\gamma_0 = 1$), then $\tilde{\boldsymbol{u}}(x)$ satisfies the hypothesis of Theorem 2 where*

$$\delta_V := \min_{x \in V, \lambda \in \Lambda(x)} J^\star(x) + \gamma_0 q(x, 0) - \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i). \tag{8}$$

This corollary follows from Remark 3, convexity of $q(x, u)$ and (7), as the following is true for all $x \in V$:

$$\delta_V \leq J^\star(x) + \gamma_0 q(x, \tilde{u}_0(x)) - J(x, \tilde{\boldsymbol{u}}(x)).$$

Definition 7 (Worst-case certificate) *We call the stability certificate established by obtaining a positive δ_V, the worst-case certificate, as stability is guaranteed for arbitrary choice of barycentric coordinates [14].*

Remark 8 *Motivation for the worst-case certificate is convexity of the optimization problem, defined in (8). However, the gap between the worst-case certification and a particular choice of barycentric functions may lead to conservatism. Consequently, the first goal in Problem 1 can be restated as proposing a certification procedure and barycentric function that satisfies Theorem 2 with lower conservatism than the worst-case certificate introduced in Definition 7.*

3 LP barycentric coordinates and a new stability certificate

In this section we introduce a novel way to compute barycentric coordinates by means of linear programming along with a stability test, which is less conservative than the worst-case test.

3.1 LP barycentric coordinates

Consider a set of points $v^i \in \mathbb{R}^n$ for $i \in \mathcal{I} = \{1, \ldots, d\}$, its convex hull V and assume that (5) is feasible for all v^i, i.e., there exist a minimizer $\boldsymbol{u}^\star(v^i)$ for the problem initialized at $x = v^i$. We introduce a barycentric function $\lambda^\star(x)$, which is defined as an optimizer of the linear program:

$$\lambda^\star(x) = \arg\min_{\lambda \in \Lambda(x)} \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i). \tag{9}$$

Using the LP barycentric coordinates introduced above, the sub-optimal control input $\hat{\boldsymbol{u}}(x)$ is proposed:

$$\hat{\boldsymbol{u}}(x) = \sum_{i \in \mathcal{I}} \lambda_i^\star(x) \boldsymbol{u}^\star(v^i). \tag{10}$$

Note that $\lambda^\star(x)$ is evaluated online by solving the corresponding linear program numerically. Luckily, its computational complexity is decoupled from system dynamics, constraints and implementation of partitioning and certification procedure.

3.2 Minimax stability certificate

The following theorem forms the basis for a new stability certificate, which takes advantage of a tight upper bound on the total cost of the sub-optimal control law to construct a less conservative test than the one based on the worst-case certificate:

Theorem 9 *Consider a set of points $v^i \in \mathbb{R}^n$ for $i \in \mathcal{I} = \{1, \ldots, d\}$, its convex hull V and an approximate solution, given by (10), of the multiparametric problem, arising from (5). Define Δ_V as follows:*

$$\Delta_V := \min_{x \in V} \max_{\lambda \in \Lambda(x)} J^\star(x) + \gamma_0 q(x, 0) - \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i). \tag{11}$$

If $\Delta_V \geq 0$, then the following is true for all $x \in V$:

$$J(x, \hat{\boldsymbol{u}}(x)) \leq J^\star(x) + \gamma_0 q(x, \hat{u}_0(x)).$$

PROOF. The first two terms on the right hand side of (11) are not functions of λ, thus:

$$\min_{x \in V} \max_{\lambda \in \Lambda(x)} \left[J^\star(x) + \gamma_0 q(x, 0) - \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i) \right] = \min_{x \in V} \left[J^\star(x) + \gamma_0 q(x, 0) - \min_{\lambda \in \Lambda(x)} \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i) \right].$$

Since the control law is defined as the convex combination, we use the convexity bound on $J(x, \hat{\boldsymbol{u}}(x))$:

$$J(x, \hat{\boldsymbol{u}}(x)) \leq \min_{\lambda \in \Lambda(x)} \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i).$$

Thus, from the convexity of $q(x, u)$:

$$\Delta_V \leq J^\star(x) + \gamma_0 q(x, \hat{u}_0(x)) - J(x, \hat{\boldsymbol{u}}(x)).$$

If $\Delta_V \geq 0$, then $\hat{\boldsymbol{u}}(x)$ meets conditions outlined in the hypothesis of Theorem 2. □

Remark 10 *It is worth noting that the inequalities*

$$J(x, \hat{\boldsymbol{u}}(x)) \leq \min_{\lambda \in \Lambda(x)} \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i) \leq \max_{\lambda \in \Lambda(x)} \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i)$$

hold with equality only if $J(x, \boldsymbol{u})$ is a linear function of its arguments, i.e.,

$$J\big(\sum_{i \in \mathcal{I}} w_i(x) v^i, \sum_{i \in \mathcal{I}} w_i(x) \boldsymbol{u}^\star(v^i)\big) = \sum_{i \in \mathcal{I}} w_i(x) J\big(v^i, \boldsymbol{u}^\star(v^i)\big).$$

Remark 11 *Since $\Delta_V \geq \delta_V$, there exists circumstances where $\Delta_V \geq 0$ for $\lambda_i^\star(x)$, while $\delta_V < 0$.*

3.3 Algorithm for computing Δ_V

Consider (9) as a multiparametric linear program:

$$f(x) = \min_{\lambda \in \Lambda(x)} \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i). \tag{12}$$

Note that $f(x)$ is convex and piece-wise affine in x (see [8] for details), so computing Δ_V using (11) requires minimization of the difference of two convex functions. It is not a convex problem. However, it can be split into convex optimization problems as described below.

Theorem 12 *Consider a set of points $v^i \in \mathbb{R}^n$ for $i \in \mathcal{I} = \{1, \ldots, d\}$, its convex hull V and assume (5) is feasible for all v^i. Consider the minimax problem:*

$$\Delta_V = \min_{x \in V} \max_{\lambda \in \Lambda(x)} \left[J^\star(x) + \gamma_0 q(x, 0) - \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i) \right].$$

There exist a partition of V into a finite number of convex regions V_j, where $j = 1, \ldots, p$, such that the problems

$$\Delta_{V_j} = \min_{x \in V_j} \left[J^\star(x) + \gamma_0 q(x, 0) - \sum_{i \in \mathcal{I}} \lambda_i^\star(x) J^\star(v^i) \right],$$

are convex and $\Delta_V = \min(\Delta_{V_1}, \ldots, \Delta_{V_p})$.

PROOF. We can transform the original problem into an equivalent one, as its first two terms are not functions of λ:

$$\min_{x \in V} \max_{\lambda \in \Lambda(x)} \left[J^\star(x) + \gamma_0 q(x, 0) - \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i) \right] = \min_{x \in V} \left[J^\star(x) + \gamma_0 q(x, 0) - \min_{\lambda \in \Lambda(x)} \sum_{i \in \mathcal{I}} \lambda_i J^\star(v^i) \right].$$

Now consider a multiparametric linear program (12). It is known that $f(x)$ is a continuous convex piece-wise affine function of parameter $x \in V$ (see [8] for details). We select regions V_j to be convex regions, where $f(x)$ is purely affine, such that their union is V, and denote with $f_j(x)$ the affine piece of $f(x)$ over the region V_j. In this way, the resulting optimization problems

$$\min_{x \in V_j} \left[J^\star(x) + \gamma_0 q(x, 0) - f_j(x) \right]$$

are convex. As the objective function is continuous, it reaches its minimum value inside one of the regions V_j. By comparing values Δ_{V_j} for all $j = 1, \ldots, p$, the value Δ_V is found to be the smallest among them. □

Remark 13 *The test based on Theorem 9 is less conservative than the worst-case certificate. However, it incurs a higher offline computational cost, as it requires solving a multiparametric linear program together with a number of convex programs.*

3.4 Augmented minimax certificate

So far we used a convex upper bound on $J(x, \hat{\boldsymbol{u}}(x))$ over the whole V. Now lets provide a separate bound for each of the regions V_j (for information on V_j see the proof of Theorem 12). As $J(x, \hat{\boldsymbol{u}}(x))$ is a sum of convex functions, linked to each other with the linear dynamics and the piece-wise affine control law. So, we can provide a tighter non-affine upper-bound for each V_j as introduced next.

Definition 14 (Augmented bound) *Define* $J_\alpha(x, \hat{\boldsymbol{u}}(x))$ *for some* $\alpha = 0, \ldots, N-1$ *as follows:*

$$J_\alpha(x, \hat{\boldsymbol{u}}(x)) = \sum_{k=0}^{\alpha-1} q(\xi_k(x), \hat{u}_k(x)) + \sum_{i \in \mathcal{I}} \lambda_i^\star(x) \Big[\sum_{k=\alpha}^{N-1} q(\xi_k(v^i), \hat{u}_k(v^i)) + p(\xi_N(v^i)) \Big],$$

where $\xi_{k+1}(x) = A\xi_k(x) + B\hat{u}_k(x)$ *(with* $\xi_0(x) = x$*) are states predicted by the linear dynamics under the control law, given by* $\hat{\boldsymbol{u}}(x)$.

Now we can try to use $J_\alpha(x, \hat{\boldsymbol{u}}(x))$ instead of $\sum_i \lambda_i^\star J^\star(v^i)$ in the certification procedure. We call the following result *α-augmented minimax certificate*, where we keep all cost terms up to the α-th explicitly and bound the remaining terms with a suitable convex combination.

Theorem 15 *Consider a set of points* $v^i \in \mathbb{R}^n$ *for* $i \in \mathcal{I} = \{1, \ldots, d\}$*, its convex hull* V *and assume (5) is feasible for all* v^i*. Consider also the control law, defined by (10), and denote by* V_j *(*$j = 1, \ldots, p$*) the regions, where the control law is affine in* x*. Define* $\Delta_{V_j}^\alpha$ *as follows:*

$$\Delta_{V_j}^\alpha := \min_{x \in V_j} \left[J^\star(x) + \gamma_0 q(x, \hat{u}_0(x)) - J_\alpha(x, \hat{\boldsymbol{u}}(x)) \right]. \quad (13)$$

If for every $j = 1, \ldots, p$ *corresponding value* $\Delta_{V_j}^\alpha \geq 0$ *for some* $\alpha = 0, \ldots, N-1$*, then* $\forall x \in V$ *the following holds:*

$$J(x, \hat{\boldsymbol{u}}(x)) \leq J^\star(x) + \gamma_0 q(x, \hat{u}_0(x)).$$

PROOF. Consider a region V_j with the affine control law, defined by (10). As the system is linear, we notice that $J(x, \hat{\boldsymbol{u}}(x))$ is a sum of terms, which are convex in x. If only a part of the terms is bounded by the convex combination, then

$$J(x, \hat{\boldsymbol{u}}(x)) \leq J_\alpha(x, \hat{\boldsymbol{u}}(x)) \leq \sum_{i \in \mathcal{I}} \lambda_i^\star J^\star(v^i).$$

Thus, if for every j, $\Delta_{V_j}^\alpha \geq 0$, then the inequality $J(x, \hat{\boldsymbol{u}}(x)) \leq J^\star(x) + \gamma_0 q(x, \hat{u}_0(x))$ holds $\forall x \in V$. □

Remark 16 *The optimization problem in Theorem 15 is guaranteed to be convex for* $\alpha = 0$*, as* $J_0(x, \hat{\boldsymbol{u}}(x))$ *is affine in* x*, while the function* $q(x, \hat{u}_0(x))$ *remains convex inside each* V_j*. Interestingly, in the case of* $\gamma_0 = 1$ *resulting program for* 1*-augmented test is guaranteed to be convex too, as the terms with* $q(x, \hat{u}_0(x))$ *cancel out:*

$$\Delta_{V_j}^1 = \min_{x \in V_j} \Big[J^\star(x) + q(x, \hat{u}_0(x)) - q(x, \hat{u}_0(x)) - \sum_{i \in \mathcal{I}} \lambda_i^\star(x) \big[J^\star(v^i) - q(v^i, \hat{u}_0(v^i)) \big] \Big].$$

Remark 17 *A brief discussion of the worst-case complexity of the off-line computation required for the proposed certification process is given below. As shown in Theorem 12, using hyper-rectangle partitions in the proposed stability test requires the solution of one multiparametric problem* (12) *and some number (denote it* p*) of convex problems for each hyper-rectangle. Hyper-rectangle* V *should be partitioned into* p *pairwise disjoint regions of non-zero volume, where the barycentric coordinates* $\lambda^\star$ *are defined as an affine function of* x *such that the minimum of problem* (12) *is obtained. At most it requires* $n!$ *simplices (where each vertex of a simplex is a vertex of the hyper-rectangle).*

4 An anisotropic partitioning strategy

We propose a partitioning strategy based on the orthogonal approach introduced in [14]. However, instead of partitioning the uncertified hyper-rectangle into 2^n hyper-rectangles, we follow an anisotropic approach and partition it along one edge at a time as explained below. Consider a Taylor series of a piece-wise smooth function $J^\star(x) : \mathbb{R}^n \to \mathbb{R}$ around $x = v$, where v is a geometrical center of the hyper-rectangle, i.e.,

$$J^\star(x) \approx J^\star(v) + \nabla J^\star(v)^T (x-v) + \frac{1}{2}(x-v)^T H(v)(x-v),$$

where $\nabla J^\star(v)$ and $H(v)$ are the gradient and Hessian of $J^\star(v)$. We also assume that the residual is negligible, so the difference between the function and its first order Taylor approximation is mostly captured by the Hessian.

Let e^j for $j = 1, \dots, n$ be the unit vectors representing the axes of a Cartesian coordinate system. Let the length of the j-th edge of a hyper-rectangle V be denoted by l_j. We denote with v^1 the vertex, which is a lower bound on all other vertices, and enumerate vertices v^i for $i = 1, \dots, 2^n$ of the hyper-rectangle as follows:

$$v^i = v^1 + \sum_{j=1}^{n} c_{ij} l_j e^j,$$

where c_{ij} are elements of matrix C, defined such that the i-th row of C is the n-bit representation of integer $i-1$. When (5) is solved at vertices v^i of the hyper-rectangle, the vectors μ^i of Lagrange multipliers corresponding with the initial state constraint ($\xi_0 = v^i$) have the following property (by the Envelope Theorem [16]):

$$\left.\frac{\partial J^\star(x)}{\partial x}\right|_{x=v^i} = -\mu^i. \qquad (14)$$

As we limit ourselves to the orthogonal partitioning strategy, we estimate diagonal elements of the Hessian $H(v)$ of $J^\star(v)$ by taking the average of finite difference estimates along the edges:

$$\hat{H}_{jj}(v) = -\frac{1}{2^{n-1} l_j} \sum_{i=1}^{2^n} (-1)^{c_{ij}} \frac{\partial J^\star(v^i)}{\partial x_j}.$$

For example, in 3-dimensional hyper-rectangle for edges aligned with e^1 it yields:

$$\hat{H}_{11}(v) = -\frac{1}{4 l_1}(-\mu^5 + \mu^1 - \mu^6 + \mu^2 - \mu^7 + \mu^3 - \mu^8 + \mu^4)^T e^1.$$

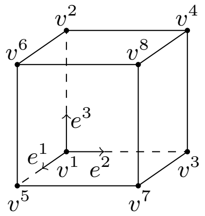

Table 1
Number of certified hyper-rectangles for different methods

System size	2	3	4	5	6
Worst-case	16	64	2296	115228	-
Anisotr. Worst-case	12	34	368	2480	9948
Anisotr. Minimax	12	20	88	436	500

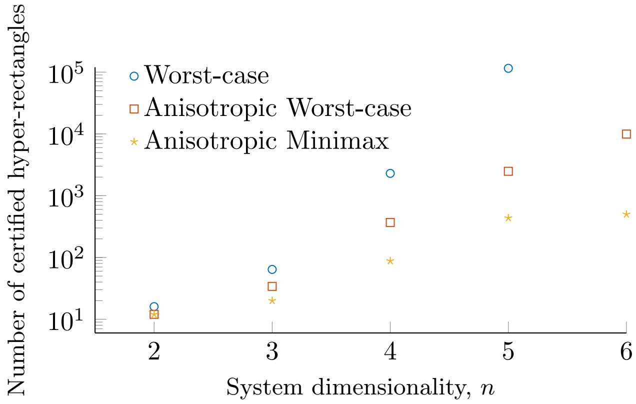

Fig. 1. Number of certified hyper-rectangles versus system dimension.

The proposed anisotropic partitioning strategy is to select the axis k with the biggest ϵ_k for further partitioning, where

$$\epsilon_k = \hat{H}_{kk}(v) l_k^2. \qquad (15)$$

5 Numerical and experimental results

In this section, first, we demonstrate scaling properties of the approaches. Second, the hardware implementation of the proposed approximate explicit MPC strategy using minimax augmented certificates and anisotropic portioning is presented. We use MPT3 [17] and YALMIP [18] for formulating and solving the problems.

5.1 Scalability of the approach

Here we generate random controllable linear systems, as in (1), with $n = 2, \dots, 6$ and one constrained input $u \in [-1, 1]$, such that the euclidean operator norms of A and B are 1. We compare worst-case certification with standard partitioning strategy, worst-case and minimax (augmented with $\alpha = 1$) certification with anisotropic partitioning strategy for $\gamma_0 = 1$. Each method is tested on the same system of given dimension for a hyper-cube $[-1; 1]^n$. We select $q(x, u) = x^T x + 0.02u^2$, $p(x)$ and $\mathcal{X}_f$ are the LQR value function and LQR control invariant set. As an illustration, for the 5-dimensional example computational time of the proposed method is about 360 minutes, while anisotropic worst-case certification took about 155 minutes.

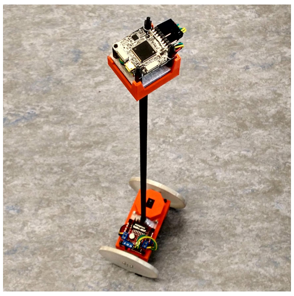

Fig. 2. Two-wheeled balancing robot

MCU	STM32F745
Inertia sensor	MPU6000
Encoders	0.69 deg
Motors torque	1.5 kgf/cm
Head mass	100 g
Base mass	500 g
Height	60 cm

Table 2. Specifications of the two-wheeled balancing robot

5.2 Hardware implementation

We consider hardware implementation of proposed strategy on a two-wheeled balancing robot[2]. The discrete time model for the robot around the unstable equilibrium with a sampling time of $0.01s$ is

$$x^+ = \begin{bmatrix} 1 & 0.01 & -0.003 & -1\text{e-}5 \\ 0 & 0.92 & -0.601 & -0.003 \\ 0 & 5\text{e-}5 & 1.001 & 0.01 \\ 0 & 0.01 & 0.222 & 1.001 \end{bmatrix} x + \begin{bmatrix} 1.2\text{e-}4 \\ 0.025 \\ -1.5\text{e-}5 \\ -0.003 \end{bmatrix} u,$$

where x is a vector of states: wheel's angle (rad) and angular velocity (rad/sec), robot's angle (rad) and angular velocity (rad/sec); input u is the duty cycle of the motor drive circuit.

We choose $q(x, u) = x^T x + 0.02u^2$ and $N = 50$. As before $p(x)$ is the LQR value function and $\mathcal{X}_f$ is the LQR control invariant set. State and input constraints are

$$-[\infty\ \infty\ \frac{\pi}{6}\ \frac{\pi}{2}]^T \leq x \leq [\infty\ \infty\ \frac{\pi}{6}\ \frac{\pi}{2}]^T,\ |u| \leq 70,$$

a hyper-rectangle to be partitioned is chosen as follows

$$-[\pi\ \pi\ \frac{\pi}{6}\ \frac{\pi}{2}]^T \leq x \leq [\pi\ \pi\ \frac{\pi}{6}\ \frac{\pi}{2}]^T.$$

The partition is obtained using the augmented minimax certificate with $\alpha = 1$, level of suboptimality $\gamma_0 = 1$, and the anisotropic partitioning strategy. After merging all hyper-rectangles with saturated input it yields: 6840 LQR invariant, 14908 minimax certified and 2056 input saturated hyper-rectangles. The position and the angle of the robot in the experiment[3] (Fig. 3) are measured by an OptiTrack camera system where the system is subjected to a disturbance at $t \approx 10s$.

In Table 3 we provide statistical properties (mean, standard deviation and 99% confidence interval) of wall clock time needed to search the the decision tree τ_{tree}, wall clock time τ_{lp} required to solve an instance of the linear program (9) via the simplex method (further evaluation of (10) takes negligible amount of time), the absolute control error $|u^\star(x) - \hat{u}(x)|$, and level of suboptimality, $\beta(x) = 1 - \frac{J^\star(x) - J^\star(x^+(\hat{u}))}{q(x,\hat{u})}$. In this example, the existing approximate explicit MPC method based on the worst-case certificate did not yield an implementable solution: the algorithm was terminated when it reached 1 million hyper-rectangles with no stability certificate. This highlights the advantage of the proposed method for practical implementations.

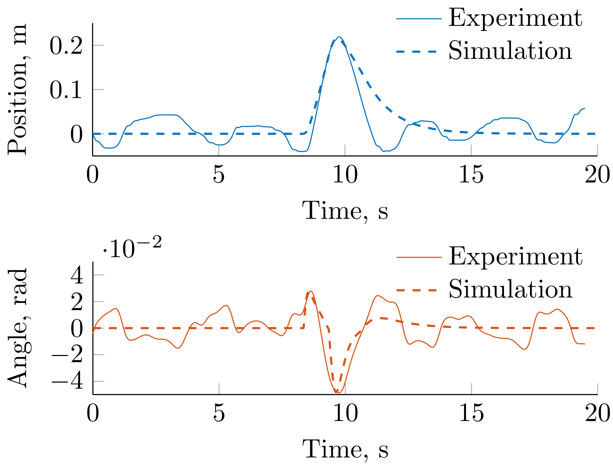

Fig. 3. Robot's position and angle during the experiment with an event at $t \approx 10$s

Table 3
Statistical properties of the control law

Variable	Mean	Standard deviation	Confidence interval
τ_{tree}, μsec	9	2	[0 14]
τ_{lp}, μsec	656	140	[0 975]
$\lvert u^\star(x) - \tilde{u}(x)\rvert$	0.5	0.9	[0 3.8]
$\beta(x)$	0.001	0.004	[0 0.017]

6 Conclusions and Future Work

We propose a new approach to approximate the Explicit MPC control law in a more efficient manner. The approach is based on a barycentric interpolation and requires offline solution of a multiparametric linear program together with a finite number of convex optimization problems to provide a certificate of stability. During the online phase only a solution of a small-scale linear program is necessary to compute barycentric coordinates. Thus, the evaluation of the control law remains computationally tractable. In addition, we propose an anisotropic approach for partitioning of uncertifiable regions, which estimates the second-order information of

[2] Link to the github project: https://git.io/fxX6A

[3] The movie of the experiment can be found at https://youtu.be/233ZM8I6WBM

the underlying problem. Case studies on low order systems have demonstrated considerable reduction in the number of partitions in comparison with a state-of-the-art approach. An example of successful implementation on a low-powered embedded system is also provided to further demonstrate the efficacy of the proposed approach. We consider further exploitation of the piecewise affine property of the proposed barycentric interpolation as a possible direction for future research.

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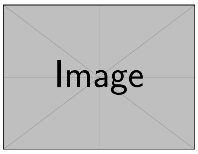

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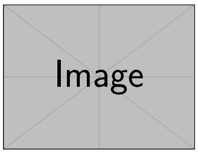

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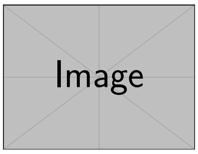

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