Complexity minimisation of suboptimal MPC without terminal constraints ⋆

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Abstract: This paper generalises stability analysis of Nonlinear Model Predictive Control without terminal constraints to incorporate possible suboptimality of MPC solutions and develops a framework for minimisation of computational efforts associated with obtaining such a solution. The framework is applied to primal-dual interior-point solvers by choosing the length of the prediction horizon together with a degree of suboptimality of the solution in a way that reduces algorithmic complexity while satisfying certain stability and performance guarantees. The framework ensures an optimal choice for the prediction horizon in order to minimise computational complexity if applied to linear or convex quadratic MPC problems, and acts as a good indicator to this end in the more general case of nonlinear systems. This is illustrated in a numerical case study, where we apply the proposed framework to a nonholonomic robot.

Keywords: Nonlinear predictive control, Real-time optimal control, Numerical methods for optimal control

1. INTRODUCTION

Model predictive control (MPC) is a closed-loop control paradigm based on online numerical optimisation, which can explicitly utilise a system model in optimising the prescribed performance metric while satisfying inherent constraints of the system over the prediction horizon. Sufficiently long prediction horizons are often desired for either enlarging the feasible region of the optimisation problem or ensuring stability and better performance. However, online optimisation of MPC problems with a long prediction horizon is often restricted due to limited computation resources available at real-time, see Richter et al. [2011] for detailed analysis.

To reduce the computational costs associated with MPC one can accept inexact solutions, which provide sufficient reduction of a Lyapunov function candidate. This was first discussed for MPC schemes with extra terminal cost and constraints, see Scokaert et al. [1999] and Diehl et al. [2005]. Alternatively, Giselsson and Rantzer [2013] consider MPC schemes without terminal constraints/cost along with a dual gradient method, and utilise the weak duality property to ensure stability and specified performance for an inexact solution. A similar early termination approach based on the the weak duality property was proposed for the primal-dual interior-point method and applied to an MPC scheme *with* stabilising terminal constraints and cost, see Pavlov et al. [2019]. Here, however, one needs no dual problem formulated, since the primal and dual problems are solved simultaneously. Moreover, satisfaction of inequality constraints is ensured by the property of the interior-point algorithm, which constructs a sequence of solutions approaching the optimal solution from the interior of the feasible region.

In this paper, we extend the stability analysis of MPC schemes *without* terminal constraints and cost, see Grüne [2012], to include possible suboptimality of a solution and admit the early termination of the solver based on the duality-gap criterion. By focusing on the primal-dual interior-point family of solvers and utilising corresponding complexity bounds, we formulate an optimisation problem to minimise the algorithmic complexity with the prediction horizon length and the degree of suboptimality as decision variables that will result in closed-loop stability.

The paper is organized as follows. In Section 2, first, the MPC framework and the idea of primal-dual interior-point methods to solve the corresponding optimisation problems in MPC are briefly introduced. Second, the problem of interest is presented. In Section 3 the main contribution of this paper is presented. In Section 4 we demonstrate a successful application of the proposed framework to a nonholonomic system. Conclusions are summarized in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

Throughout the paper we use the following classes of continuous functions, where we denote $\mathbb{R}_0^+ = [0, \infty)$:

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$$\mathcal{K} := \{\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ |\ \phi(0) = 0 \text{ and strictly increasing}\},$$
$$\mathcal{L} := \{\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ |\ \phi(\infty) = 0 \text{ and strictly decreasing}\},$$
$$\mathcal{K}_\infty := \{\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ |\ \phi \in \mathcal{K} \text{ and } \lim_{r\to\infty} \phi(r) = \infty\}, \text{ and}$$
$$\mathcal{KL} := \{\phi : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ |\ \phi(\cdot, s) \in \mathcal{K},\ \phi(r, \cdot) \in \mathcal{L}\}.$$

2.1 Model predictive control

Consider a discrete-time system of the following form:

$$x^+ = f(x, u),$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a sufficiently smooth map with $f(0, 0) = 0$, which for a current state $x \in \mathbb{R}^n$ and a control input $u \in \mathbb{R}^m$ assigns new state x^+ at the next time step.

Assume that the states and inputs of the system are subject to constraints $x \in \mathcal{X}$ and $u \in \mathcal{U}$, which contain the origin and can be written algebraically as follows:

$$c(x, u) \le 0,$$

where $c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$ is smooth and $c(0, 0) < 0$.

Define the stage cost $\ell(x, u)$ as a cost for being in state x while taking action u, and define

$$\ell^\star(x) := \min_u\ \ell(x, u), \text{ s.t. } \quad c(x, u) \le 0. \tag{1}$$

We assume that $\ell(x, u)$ is smooth and can be bounded from below and above by $\mathcal{K}_\infty$−functions $\phi_1(\cdot)$ and $\phi_2(\cdot)$ such that for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$:

$$\phi_1(\|x\|) \le \ell(x, u) \le \phi_2(\|x\|). \tag{2}$$

Consider the task of driving the system to the origin from the current state x. We formulate a finite-horizon constrained optimal control problem to be solved at each time step as follows:

$$\begin{aligned} V_N(x) = \min_{\boldsymbol{z},\boldsymbol{u}} & \sum_{i=0}^{N-1} \ell(z_i, u_i) \\ & \text{s.t. for } i = 0, \ldots, N-1 : \\ & z_0 = x,\ z_{i+1} = f(z_i, u_i), \\ & c(z_i, u_i) \le 0, \end{aligned} \tag{3}$$

where $\boldsymbol{u} = [u_0; \ldots; u_{N-1}]$ and $\boldsymbol{z} = [z_0; \ldots; z_N]$ are stacks of the vectors of decision variables, and $N \in \mathbb{N}_{\ge 2}$, where $\mathbb{N}_{\ge k}$ is the set of integers greater or equal to k.

For simplicity, we assume that $\mathcal{X}$ is control invariant, i.e., for each $x \in \mathcal{X}$ there exists $u \in \mathcal{U}$ with $f(x, u) \in \mathcal{X}$, which is often done in the context of MPC without terminal constraints. If this is not satisfied, one can use additional arguments in order to still ensure recursive feasibility on sublevel sets of the optimal value function V_N, see Boccia et al. [2014] for details.

We now repeat the main result for asymptotic stability and closed-loop performance established in Theorem 4.11 of Grüne and Pannek [2017]:

Theorem 1. Consider an MPC problem, defined by (3). Let the admissible closed-loop control law $u(x)$ be such that the following condition is satisfied for some $\alpha \in (0, 1]$ for all $x \in \mathcal{X}$:

$$V_N(f(x, u(x))) \le V_N(x) - \alpha\ell(x, u(x)). \tag{4}$$

Then the origin is asymptotically stable under the closed-loop control $u(x)$ and the following holds:

$$J_\infty(x, u(x)) \le \alpha^{-1} V_N(x),$$

where $J_\infty(x, u(x)) = \sum_{i=0}^{\infty} \ell(z_i, u(z_i))$, $z_0 = x$ and $z_{i+1} = f(z_i, u(z_i))$.

Remark 2. Since $V_N(x) \le V_\infty(x)$ we also have

$$J_\infty(x, u(x)) \le \alpha^{-1} V_\infty(x).$$

Definition 3. In the light of Remark 2 we call α the performance measure, as it relates the total cost under the closed-loop control to the infinite horizon optimal cost.

2.2 Primal-dual interior-point method

Interior-point methods are a class of algorithms for finding local optimisers of constrained optimisation problems, see Gondzio [2012] for an overview. Here we consider a primal-dual interior-point method for solving the finite-horizon constrained optimal control problems arising in the MPC framework and briefly outline its algorithmic complexity results.

The Lagrange function for problem (3) is given by:

$$L = \sum_{i=0}^{N-1} \left[\ell(z_i, u_i) + s_i^T c(z_i, u_i) + \lambda_i^T (f(z_i, u_i) - z_{i+1})\right],$$

where $\lambda_i \in \mathbb{R}^n$ and $s_i \in \mathbb{R}^l$ are the vectors of dual variables, and $\boldsymbol{z}$, $\boldsymbol{u}$, $\boldsymbol{\lambda}$, $\boldsymbol{s}$ are stacks of the corresponding vectors, e.g. $\boldsymbol{s} = [s_0; \ldots; s_{N-1}]$. We introduce the i-th l-dimensional vector of slack variables $y_i \in \mathbb{R}^l$ such that $c(z_i, u_i) + y_i = 0$ and, similarly, $\boldsymbol{y} = [y_0; \ldots; y_{N-1}]$, and denote $S = \text{diag}[\boldsymbol{s}]$ and $Y = \text{diag}[\boldsymbol{y}]$.

Once the Lagrange function is defined, the original optimisation problem can be cast as a min-max problem:

$$V_N(x) = \min_{\boldsymbol{z},\boldsymbol{u}} \max_{\boldsymbol{s}\ge 0,\boldsymbol{\lambda}} L(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}), \tag{5}$$

where the minimisation is (possibly) understood to be local in a neighbourhood of a local minimum of (3).

The idea of the primal-dual interior-point methods is to solve the system of necessary conditions for optimality (KKT) for a decaying perturbation $\mu > 0$, while keeping $\boldsymbol{s}$ and $\boldsymbol{y}$ strictly positive. Here, the perturbed KKT system is defined with $i = 0, \ldots, N-1$ as follows:

$$\begin{aligned} \nabla_{\boldsymbol{z},\boldsymbol{u}} L(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}) &= 0, \\ f(z_i, u_i) - z_{i+1} &= 0, \\ c(z_i, u_i) + y_i &= 0, \\ SYe &= \mu e, \\ (\boldsymbol{s}, \boldsymbol{y}) &> 0, \end{aligned} \tag{6}$$

where $z_0 = x$ and e is the vector of all ones, see Nocedal and Wright [2006] for more details.

A locally optimal solution of (5) can be obtained as a limit point of solutions to (6) with $\mu \to 0$. To finish the calculation in finite time the algorithm checks whether the user-provided tolerance is achieved and terminates the sequence if the second-order sufficient condition for a constrained local minimum holds.

As a single iteration of the interior point methods requires a computation of the Newton direction for the perturbed KKT system, it is necessary to compute a factorisation of a (possibly large) matrix of size d with the complexity bound $O(d^3)$ at each step. For the case of a *non-condensed* formulation of the MPC problem, where states are kept as decision variables, one can utilise the block-diagonal structure of the matrix to come up with a linear complexity bound in the prediction horizon N, e.g. $O(N)$ for Gaussian elimination technique as opposed to $O(N^3)$ if the special structure is not exploited as in a condensed formulation, see Rao et al. [1998] for details.

For linear or convex quadratic programming problems, the best known to date algorithms within the family of short-step interior-point methods, given an appropriate solution to (6) with the initial perturbation $\mu = \mu_0$, provably converge to an ϵ-accurate solution, i.e. a solution to (6) with $\mu \leq \epsilon$, in at most $O(\sqrt{d}\log(\mu_0/\epsilon))$ iterations, see Potra and Wright [2000] for further details. Here $d = Nl$ is the dimension of the problem, i.e. the total number of dual variables corresponding to N vector inequalities of size l. The user-specified accuracy level ϵ is often chosen to be sufficiently small, such that the algorithm is expected to terminate at a nearly optimal solution.

2.3 Problem Statement

To be able to state the problem of interest, we define *algorithmic complexity* of the MPC problem as the product of the number of interior-point iterations needed to reach an ϵ-accurate solution of (6) and the complexity of each iteration, and denote the complexity by $\mathrm{Comp}(\epsilon, N)$.

For example, given an initial solution to (6) for $\mu = \mu_0$ and a non-condensed formulation of a linear or convex quadratic MPC problem, the algorithmic complexity for a short-step primal-dual interior-point method is given by

$$\mathrm{Comp}(\epsilon, N) = C_0 N\sqrt{N}\log(\mu_0/\epsilon). \tag{7}$$

where C_0 is a problem- and algorithm-specific positive constant.

Problem 1. Consider the MPC problem (3) along with an interior-point method. Find a prediction horizon N and an accuracy level ϵ that minimises $\mathrm{Comp}(\epsilon, N)$, the algorithmic complexity of the MPC problem, and results in closed-loop stability of the system and a certain lower bound on its performance, as defined in Definition 3.

Problem 1 in its current form is not directly amenable to analysis. In the next section, we introduce additional results that allow us to provide a solution to this problem.

3. OPTIMAL PREDICTION HORIZON FOR MINIMISED ALGORITHMIC COMPLEXITY

In order to ensure real-time implementability one would prefer the shortest possible horizon N such that a specified performance level α is attained. For complete complexity analysis, also the required suboptimality has to be taken into account, which will be done in the following. We first introduce a standard controllability assumption in the context of MPC without terminal constraints, see Grüne and Pannek [2017].

Assumption 4. The dynamical system is asymptotically controllable with respect to the stage cost $\ell(x, u)$ with rate $\beta \in \mathcal{KL}$, i.e. for all $x \in \mathcal{X}$ and all $N \in \mathbb{N}_{\geq 1}$, there exists an admissible state-control sequence $(\hat{z}_i, \hat{u}_i)$, where $\hat{z}_0 = x$ and $\hat{z}_{i+1} = f(\hat{z}_i, \hat{u}_i)$, such that for $i = 0, \dots, N-1$ the following holds:

$$\ell(\hat{z}_i, \hat{u}_i) \leq \beta(\ell^\star(x), i),$$

where $\ell^\star(x)$ is given by (1).

Assumption 5. The function $\beta \in \mathcal{KL}$ is linear in the first argument and submultiplicative, i.e. $\beta(\ell, i_1 + i_2) \leq \beta(\beta(\ell, i_1), i_2)$ for $\ell \geq 0$ and $i_1, i_2 \in \mathbb{N}_{\geq 0}$.

Remark 6. Validity of Assumption 5 can readily be established for exponential controllability or, with a certain modification, for finite-time controllability, see Proposition 7 of Sontag [1998] for details.

Definition 7. We call the iteration $(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}, \boldsymbol{y})$ of an interior-point method a **γ-suboptimal solution** to (3) with a **degree of suboptimality** $\gamma \in [0, 1)$ if it satisfies (6) for $\mu \leq \mu_\gamma$, where

$$\mu_\gamma = \frac{\gamma}{Nl}\ell(z_0, u_0). \tag{8}$$

Assumption 8. The Hessian of the Lagrange function is positive semi-definite in a neighbourhood of a local minimum of problem (3). This neighbourhood contains the γ-suboptimal solution, given by $(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}, \boldsymbol{y})$.

Remark 9. The requirement for the Hessian to be locally positive semi-definite can be restrictive and might be difficult to check in general, but is required for the local duality theory to be valid, see Chapter 14.2 of Luenberger and Ye [2008]. Note that it always holds for convex problems.

Corollary 10. For a γ-suboptimal solution $(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}, \boldsymbol{y})$ the following is true under Assumption 8

$$\sum_{i=j}^{N-1} \ell(z_i, u_i) \leq V_{N-j}(z_j) + \gamma\frac{N-j}{N}\ell(z_0, u_0).$$

where $j = 0, \dots, N-1$.

Proof. The dual problem for (5) is defined by changing the order of min and max operators, its solution obeys the weak duality property:

$$\max_{\boldsymbol{s}\geq 0, \boldsymbol{\lambda}} \min_{\boldsymbol{z}, \boldsymbol{u}} L(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}) \leq \min_{\boldsymbol{z}, \boldsymbol{u}} \max_{\boldsymbol{s}\geq 0, \boldsymbol{\lambda}} L(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}),$$

As the stationary conditions are satisfied from the definition of the γ-suboptimal solution, for given values $\boldsymbol{s}$ and $\boldsymbol{\lambda}$ the local minimum with respect to $\boldsymbol{z}$ and $\boldsymbol{u}$ is attained:

$$L(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}) = \min_{\boldsymbol{z}, \boldsymbol{u}} L(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}) \leq \max_{\boldsymbol{s}\geq 0, \boldsymbol{\lambda}} \min_{\boldsymbol{z}, \boldsymbol{u}} L(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}).$$

Moreover, since (5) is true and $f(z_i, u_i) - z_{i+1} = 0$ holds:

$$L(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}) = \sum_{i=0}^{N-1} \ell(z_i, u_i) - s_i^T y_i \leq V_N(x),$$

where $\sum_{i=0}^{N-1} s_i^T y_i = Nl\mu \leq \gamma\ell(z_0, u_0)$.
Similarly, for the tails of $(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}, \boldsymbol{y})$ for $j = 1, \dots, N-1$ we establish:

$$\sum_{i=j}^{N-1} \ell(z_i, u_i) - s_i^T y_i \leq V_{N-j}(z_j).$$

$\square$

Now we define functions B_k as follows

$$B_k(\ell) := \sum_{i=0}^{k-1} \beta(\ell, i) \overset{(*)}{=} \nu_k \ell, \tag{9}$$

where $(*)$ is true, for some positive constants ν_k, under Assumption 5.

Proposition 11. Consider a γ-suboptimal solution, denoted by $(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}, \boldsymbol{y})$, and let Assumption 4 be satisfied. Then the following upper bounds for $j = 1, \dots, N-1$ hold:

$$V_N(z_1) \leq \sum_{i=1}^{j} \ell(z_i, u_i) + B_{N-j}(\ell^\star(z_{j+1})).$$

Proof. To show validity of the proposition one can bound the optimal cost by a sum of stage costs generated by the following suboptimal control trajectory:

$$u_1, \ldots, u_j, \hat{u}_{j+1}, \ldots, \hat{u}_N,$$

where $u_1, \ldots, u_j$ is obtained from the γ-suboptimal solution and $\hat{u}_{j+1}, \ldots, \hat{u}_N$ is an admissible control sequence, which exists according to Assumption 4. The result then follows using the definition of B_k in (9). □

Proposition 12. Consider a γ-suboptimal solution to (3) for a given state x, denoted by $(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}, \boldsymbol{y})$, with $\gamma \in [0; 1)$, and let Assumptions 4 and 8 be satisfied.

If there exists $\alpha > 0$ such that for all $x \in \mathcal{X}$ the sum of stage costs satisfies

$$\sum_{i=0}^{N-1} \ell(z_k, u_k) \geq V_N(z_1) + (\alpha + \gamma)\ell(z_0, u_0),$$

then the control law generated by the γ-suboptimal solution, i.e. $u(x) = u_0$, satisfies the hypothesis of Theorem 1.

Proof. Since Assumption 4 holds, by Collorary 10 and the hypothesis of the proposition we have:

$$\sum_{i=0}^{N-1} \ell(z_k, u_k) \leq V_N(z_0) + \gamma\ell(z_0, u_0),$$

$$\sum_{i=0}^{N-1} \ell(z_k, u_k) \geq V_N(z_1) + (\alpha + \gamma)\ell(z_0, u_0),$$

and hence the hypothesis of Theorem 1 is satisfied:

$$V_N(z_1) \leq V_N(z_0) - \alpha\ell(z_0, u_0).$$

□

Theorem 13. Consider a γ-suboptimal solution, denoted by $(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}, \boldsymbol{y})$, and the following optimisation problem:

$$\begin{aligned} \alpha^\star = \min_{\ell_0, \ldots, \ell_{N-1}, v} \quad & \Big[\frac{\sum_{i=0}^{N-1} \ell_i - v}{\ell_0} - \gamma\Big] \\ \text{subject to} \quad & (\ell_0, \cdots, \ell_{N-1}, v) \in \mathcal{C} \end{aligned} \tag{10}$$

where

$$\begin{aligned} \mathcal{C} := \{(\ell_0, \cdots, \ell_{N-1}, v) \geq 0 \mid \text{for } j = 0, \ldots, N-2 : \\ \sum_{i=j}^{N-1} \ell_i \leq B_{N-j}(\ell_j) + \gamma\frac{N-j}{N}\ell_0, \\ v \leq \sum_{i=1}^{j} \ell_i + B_{N-j}(\ell_{j+1})\}. \end{aligned} \tag{11}$$

If the optimal value, $\alpha^\star$, is strictly positive, then for the γ-suboptimal solution the hypothesis of Proposition 12 is satisfied.

Proof. Consider a γ-suboptimal solution $(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{s}, \boldsymbol{y})$ with $\gamma \in [0; 1)$. Choose $\ell_i = \ell(z_i, u_i)$ and $v = V_N(z_1)$, which are non-negative by definition. These choices of $\ell_0, \ldots, \ell_{N-1}, v$ satisfy constraints (11) by Corollary 10 and Proposition 11, as $V_{N-j}(z_j) \leq B_{N-j}(\ell_j)$. Next, we assume that $\ell_0 > 0$, as the hypothesis of Proposition 12 is trivially satisfied if $\ell_0 = 0$, since $z_0 = 0$ by (2) in this case.

As $\alpha^\star$ is the minimal value of the objective function:

$$\frac{\sum_{i=0}^{N-1} \ell_i - v}{\ell_0} - \gamma \geq \alpha^\star,$$

which by multiplying with ℓ_0 gives

$$\sum_{i=0}^{N-1} \ell_i \geq v + (\alpha^\star + \gamma)\ell_0.$$

Since $\alpha^\star > 0$ the hypothesis of Proposition 12 is satisfied. □

Proposition 14. Let Assumption 5 be satisfied, then the optimal value of the optimisation problem (10) is given by

$$\alpha^\star = 1 - \gamma - \frac{\nu_N + \gamma - 1}{\prod_{i=2}^{N} \frac{\nu_i}{\nu_i - 1} - 1}, \tag{12}$$

where ν_i for $i = 2, \ldots, N$ are the constants from (9).

Proof. (sketch) The complete proof of this result is rather long and technical. In the following, we will outline the modifications which are needed compared to the proof of Proposition 6.17 in Grüne and Pannek [2017] for $\gamma = 0$ to arrive at the required result.

We start by noting that imposing $\ell_0 = 1$ in (10) is equivalent to scaling of all $\ell_0, \ldots, \ell_{N-1}, v$ by a constant and does not affect the objective function or constraints, thus the problem is equivalent to

$$\begin{aligned} \alpha^\star + \gamma = \min_{\ell_0, \ldots, \ell_{N-1}, v} \quad & \Big[\sum_{i=0}^{N-1} \ell_i - v\Big] \\ & \text{subject to } (\ell_0, \cdots, \ell_{N-1}, v) \in \mathcal{C},\ \ell_0 = 1. \end{aligned} \tag{13}$$

where γ was moved to the left side as it is a fixed constant. This is a modified version of the problem given by equation (6.17) of Grüne and Pannek [2017]: it has the same objective function, but perturbed inequality constraints, which however does not affect the arguments used to choose a set of active constraints:

$$\sum_{i=0}^{N-1} \ell_i = \nu_N + \gamma \text{ and } v = \sum_{i=1}^{j} \ell_i + \nu_{N-j}\ell_{j+1},$$

where $j = 1, \ldots, N-2$, and the system of N linear equations in N variables can be solved. However, using linearity we can get the final result by just replacing ν_N in the numerator with $\nu_N + \gamma$ in the formula with $\gamma = 0$ from Grüne and Pannek [2010]:

$$\min_{(\ell_0, \ldots, \ell_{N-1}, v) \in \mathcal{C},\ \ell_0 = 1} \Big[\sum_{i=0}^{N-1} \ell_i - v\Big] = 1 - \frac{\nu_N - 1}{\prod_{i=2}^{N} \frac{\nu_i}{\nu_i - 1} - 1},$$

and write

$$\alpha^\star + \gamma = 1 - \frac{\nu_N + \gamma - 1}{\prod_{i=2}^{N} \frac{\nu_i}{\nu_i - 1} - 1}.$$

□

Proposition 14 provides an analytical expression for the lower bound on MPC performance in the sense of Definition 3. If (6) is solved for $\mu = \epsilon$, where $\epsilon = \frac{\gamma}{Nl}\ell^\star(x)$, then $\mu \leq \mu_\gamma$, since $\ell^\star(x) \leq \ell(x, u)$ (see (1) and (8)), and thus the solution is a γ-suboptimal solution. Problem 1 can now be modified as follows.

Problem 2. Consider the MPC problem (3) along with an interior-point method. Given a positive scalar α_{min}, find a degree of suboptimality $\gamma \in [0, 1)$ and the prediction horizon N such that the algorithmic complexity $\mathrm{Comp}(\epsilon, N)$

with $\epsilon = \frac{\gamma}{Nl}\ell^\star(x)$ is minimised and the constraint $\alpha^\star \geq \alpha_{min}$ is satisfied, where $\alpha^\star$ is defined by (12).

For a path-following interior point method and convex quadratic problems, Problem 2 can be cast as follows

$$\begin{aligned} \min_{N\geq 2, \gamma\in[0,1)} \quad & N^{1.5}\log\left(\frac{Nl\mu_0}{\gamma\ell^\star(x)}\right) \\ \text{s.t.} \quad & 1-\gamma-\frac{\nu_N+\gamma-1}{\prod_{i=2}^{N}\frac{\nu_i}{\nu_i-1}-1} \geq \alpha_{min} \end{aligned} \tag{14}$$

where $\ell^\star(x)$ is given by (1) and x is the current state of the system.

Note that for each N the objective function is strictly decreasing in γ, thus the inequality constraint is always active at the optimal solution (given a feasible solution $\gamma \in [0, 1)$ exists), i.e.,

$$\gamma^\star(N) = 1 - \alpha_{min} + \frac{\alpha_{min}-\nu_N}{\prod_{i=2}^{N}\frac{\nu_i}{\nu_i-1}}.$$

This optimisation problem involves the discrete decision variable N, and one possible way to solve it is to start with $N = 2$, compute and check whether $\gamma^\star(N) \in [0; 1)$, increment N and repeat. Since the objective function is convex in N for all $N \geq \bar{N}$, where

$$\bar{N} = \frac{\gamma\ell^\star(x)}{l\mu_0}e^{-\frac{2}{3}},$$

the process is terminated when $\gamma^\star(N) \in [0, 1)$ and the value of the objective function can not be reduced by increasing N.

Although no theoretical guarantees for an optimal choice of N in terms of complexity minimisation can be given if (14) is applied to the analysis of nonlinear MPC, we show in the following section that it still can serve as a good indicator for a good choice of N. In any case, the preceding analysis shows that if N is chosen according to (14), the closed-loop system is asymptotically stable in the (mildly) nonlinear case.

4. EXAMPLE: NONHOLONOMIC ROBOT

In this section, we apply the proposed framework to a nonholonomic robot with the following kinematic model:

$$x^+ = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + T\begin{bmatrix} u_1 \operatorname{sinc}(u_2T/2)\cos(x_3+u_2T/2) \\ u_1 \operatorname{sinc}(u_2T/2)\sin(x_3+u_2T/2) \\ u_2 \end{bmatrix}$$

where $x = [x_1; x_2; x_3]$ and $u = [u_1; u_2]$ are state and input vectors, which are subject to the following constraints: $x \in [-2, 2]^2 \times \mathbb{R}$ and $u \in [-0.6, 0.6] \times [-\pi/4, \pi/4]$, and T is a sampling time. The stage cost is chosen as follows:

$$\ell(x, u) = q_1x_1^4 + q_2x_2^2 + q_3x_3^4 + r_1v^4 + r_2w^4.$$

For this choice of $\ell(x, u)$ one can ensure satisfaction of Assumptions 4 and 5 by constructing suitable admissible state and control trajectories. We use expressions for ν_i for $i = 1, \ldots, N$ provided in Worthmann et al. [2015], and note that the expressions rely on the partitioning of the state space into two regions:

$$x \in \mathbb{R}^3 : \ell^\star([x_1; x_2; 0]) < \rho \text{ and } \ell^\star([x_1; x_2; 0]) \geq \rho,$$

with an additional tunable parameter ρ, which helps to mitigate the nonoptimality of manoeuvres to some degree.

Further we fix $x = [0; 1; 0]^T$, $q_1 = 1$, $q_3 = 0.1$, $r_1 = q_1T/2$, $r_2 = q_3T/2$ and consider combinations of $q_2 = 2, 5, 10, 100$ and $T = 1, 0.5, 0.25, 0.1$. In the complexity minimisation problem (14) we choose $\alpha_{min} = 0.1$, and also include the aforementioned parameter ρ as an additional decision variable. Furthermore, we consider $\mu_0 = \ell^\star(x)$. This choice is a heuristic (and randomly generated initial guesses for a solution of (3), which we will use later, might not satisfy the perturbed KKT system with this μ_0), which ensures that the objective function in (14) is independent of x. Note that this choice of μ_0 is not justified if the initial guess is constructed using the solution from the previous time step.

Optimal horizon lengths $N^\star$ and corresponding optimal degrees of suboptimality $\gamma^\star$ are provided in Table 1 (and will be denoted as circles on the figures later in the section). The optimal values for the degree of suboptimality (for $q_2 = 2, 5$ and $T = 0.5, 0.25$) are plotted in Fig. 1, where $\gamma^\star(N)$ corresponds to the minimiser of (14) for a given (fixed) N. The minimal values of N such that the problem is still feasible with $\gamma > 0$ are found to be 21, 26 for $q_2 = 5$, $q_2 = 2$ ($T = 0.5$) and 41, 52 for $q_2 = 5$, $q_2 = 2$ ($T = 0.25$). As can be seen from Fig. 2, the optimal values of N in terms of computational complexity are higher than the minimal one required for asymptotic stability (i.e., for feasibility of (14) with $\gamma > 0$).

Table 1. Optimal values for the length of prediction horizon and degree of suboptimality

T	q_2=2	q_2=5	q_2=10	q_2=100
1	14, 0.125	11, 0.113	10, 0.104	9, 0.060
0.5	28, 0.087	22, 0.082	20, 0.082	19, 0.101
0.25	55, 0.064	43, 0.071	40, 0.073	37, 0.072
0.1	140, 0.065	107, 0.061	97, 0.059	91, 0.065

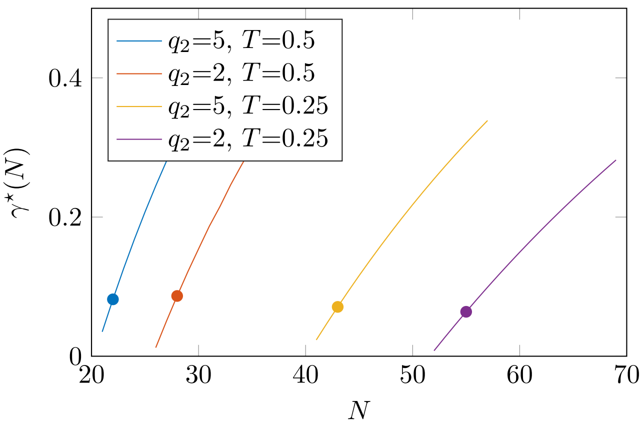

Fig. 1. Optimal values for the degree of suboptimality

To demonstrate the applicability of the proposed framework, we solve the MPC problem for the given system at $x = [0; 1; 0]^T$ multiple times, initialising the solver with a random initial guess, and measure the computational efforts as a product of the number of iterations and the prediction horizon N. We use a nonlinear primal-dual interior-point solver, namely IPOPT, see Wächter and Biegler [2006] for details on its implementation, to measure the number of iterations required to solve the MPC problem to the desired degrees of suboptimality $\gamma^\star(N)$. The optimal values of $\mathrm{Comp}(\gamma^\star(N), N)$ for different prediction horizon lengths are provided in Fig. 2, and the mean values and standard deviations of the computational efforts during the experiment are plotted in Fig. 3.

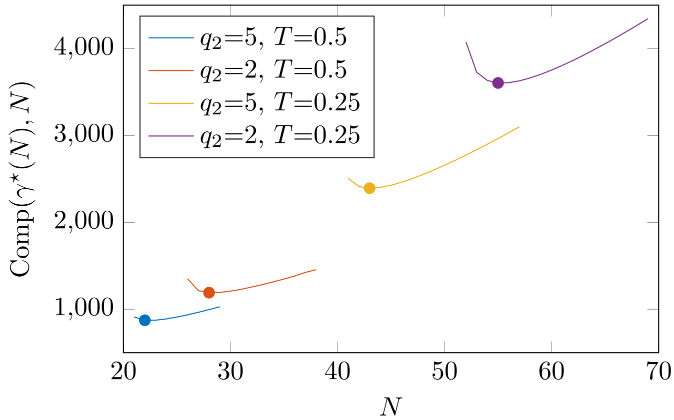

Fig. 2. Computational complexity over prediction horizon length for different system parameters.

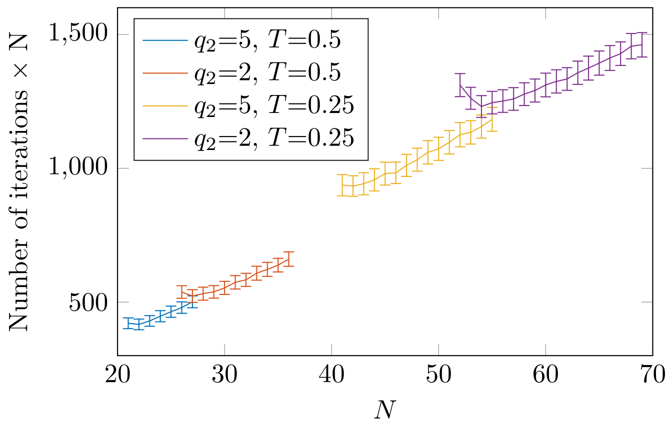

Fig. 3. Measured computational efforts over prediction horizon length for different system parameters.

While the upper bound on the number of iterations (up to multiplication by a constant), utilised in (14), was established for linear systems only, the shape of the measured curves still resembles the ones obtained by means of analysis, as is evident from Fig. 2 and Fig. 3. Hence, although optimality of $N^\star$, computed via (14), can not be guaranteed, these values are close to the prediction horizon lengths, which minimise algorithmic complexity in the numerical experiment. This means that computing $N^\star$ via (14) is well suited also for this nonlinear example. Furthermore, note that, as discussed in the previous section, the choice of $\gamma^\star$ preserves stability, despite the lack of optimality guarantee in terms of minimising computational complexity.

5. CONCLUSIONS

We proposed a framework for algorithmic complexity minimisation for a family of MPC solvers by incorporating suboptimality considerations into the MPC stability analysis. The framework is to be used during the design and verification of MPC schemes without terminal constraints with the interior-point method chosen as a solver. While rigorous guarantees on an optimal choice of the prediction horizon length have been derived for convex problems, a numerical example demonstrates that the framework and its outcomes remain applicable even when the practical implementation of long-step primal-dual interior point algorithm, namely IPOPT, is used for solving a nonlinear MPC problem. In particular, our main result shows that one might decrease computational efforts associated with MPC control by increasing the prediction horizon length above the minimum stabilising value and scheduling early termination of the algorithm at a suboptimal solution.

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